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CRI: Procedural Method for Pulley Kinematic Equations

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CRI: A Novel Procedural Approach to Pulley Kinematic Equations

Abstract

Frequently engineering dynamics courses and textbooks introduce the concept of dependent motion applied to pulley kinematics with simple examples and a single theoretical concept: the length of a rope does not change. Students are left with a challenging generalized procedure to apply to unique and novel pulley scenarios. This classroom ready innovation (CRI) provides a procedural and algorithmic approach to pulley kinematics that is easy to explain, remember, and apply. Traditional derivation and pulley presentation are documented alongside a novel approach to analyzing pulley kinematics. By using the three languages of engineering, students describe the pulley system with words, sketch the pulley system including sign convention and rope count, and write mathematical expressions describing change in position, velocity and acceleration. When coupled with other kinematic relationship equations, students then solve for the kinematic properties of various moving points in a pulley system.

Keywords

dynamics, pulleys, kinematics, dependent motion, classroom ready

Notation

$a_{A/B} =$	the relative acceleration of point *A* with respect to point *B*
$a_i =$	the acceleration of a moving point in a pulley system
$a_{is} =$	the second derivative with respect to time of a rope segment of variable length; also known as the relative acceleration of one moving point with respect to another point in a pulley system
$L_{const} =$	The constant total length in a pulley system that can be redistributed between moving points in that system
$L_i =$	the length of a rope segment that remains constant, no matter the pulley system configuration, e.g. the rope lengths between fixed points and rope lengths in contact with each pulley
$L_{rope} =$	the total length of a particular rope in a pulley system
$s_{A/B} =$	the relative position of point *A* with respect to point *B*
$s_i =$	the absolute position of a moving point in a pulley system
$s_{is} =$	the length of a rope segment that varies based on pulley system configuration; also known as the relative position of one moving point with respect to another point in a pulley system
$v_{A/B} =$	the relative velocity of point *A* with respect to point *B*
$v_i =$	the velocity of a moving point in a pulley system
$v_{is} =$	the rate of change (or the first derivative with respect to time) in length of a segment of rope; also known as, the relative velocity of one moving point with respect to another point in a pulley system

Introduction

Learning one-dimensional dependent motion in the form of pulley kinematics as part of an engineering dynamics course can challenge students, leaving them confused and frustrated. Some textbooks barely acknowledge dependent motion, simply reminding readers that they should consider physical connections between moving particles and do not explicitly acknowledge pulleys at all [1], [2]. Other textbooks describe dependent motion, but only address pulley kinematics through homework problems [3] or example problems [4], [5]. Some textbooks do introduce pulley systems as a distinct topic with varying levels of clarity and robustness [6], [7], [8]. However, the explanation for pulley system kinematics fundamentally reduces to "the length of the rope connecting pulleys does not change". Students read these presentations but frequently find themselves still confused and ill-equipped to develop meaningful equations to describe the kinematics of pulley systems. A novel procedural approach to developing pulley kinematic relationship equations described here-in compliments the traditional derivation and development of pulley system equations as found in existing textbooks.

The first section of this paper presents three methods for developing pulley kinematic relationship equations. All pulley kinematic relationship equations are based on the inextensible and unchanging length of the rope(s) in a pulley system. A theoretical derivation written from first principles and relative motion equations shows how working like a physicist can result in complex but comprehensive systems of equations [7], [8]. The second traditional presentation starts with the same basic claim (the rope length does not change) but describes the length of the rope in terms of the distances from fixed reference points to moving points of interest [4], [5], [6], [7]. The third novel presentation, a classroom ready innovation (CRI), builds off the traditional presentation but develops the dependent kinematic relationship equations using an algorithmic procedure based on a standard sign convention, rope count coefficients, and generalized equations. The CRI section includes the learning objectives and activities for teaching pulley kinematics in the classroom.

Three Approaches to Developing Dependent Motion Kinematic Relationship Equations for Pulley Systems

Theoretical Development

The theoretical development of the pulley kinematic equations involves labeling the lengths of all rope segments on an example pulley system such as in Figure 1. The length of the rope equals the sum of all the labeled segment lengths (Equation 1). Several of the rope segments will remain constant in length due to their connection to fixed points or their contact with pulley arcs (labeled using L in Figure 1). Other rope segments will redistribute and change in length as the points move (labeled using s in Figure 1), resulting in a dependent relationship between moving point positions. In Equation 2, all the constant lengths are collected on the left side of the equal sign, leaving the variable lengths on the right side of the equation. The constant lengths can be simplified to a singular constant as shown in Equation 3. This is the fundamental equation that, in this case, describes the relationship between point *A* and point *C* of the example in Figure 1.

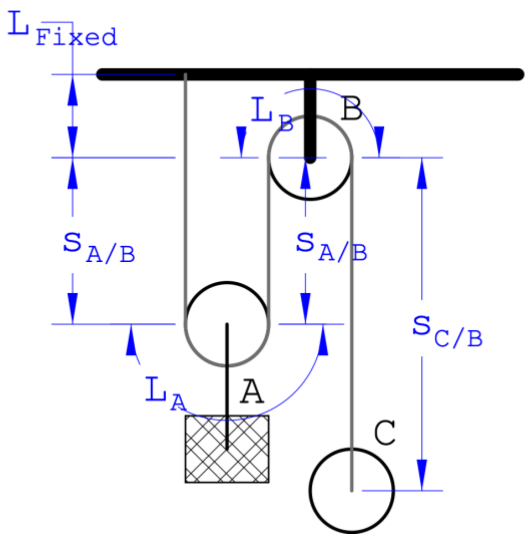


Figure 1. Example pulley system with labeled rope lengths: as the moving points change position, segments labeled L remain constant in length, and segments labeled s vary in length.

$$L_{rope} = L_{Fixed} + s_{A/B} + L_A + s_{A/B} + L_B + s_{C/B} \tag{1}$$

$$L_{rope} - L_{Fixed} - L_A - L_B = 2\, s_{A/B} + s_{C/B} \tag{2}$$

$$L_{const} = 2\, s_{A/B} + s_{C/B} \tag{3}$$

If the pulley system changes from an initial position (subscript *o*) to a final position (subscript *f*), Equation 3 can describe both instances as shown in Equation 4. By subtracting the initial positions from the final positions (Equation 5), the useful Equation 6 references the relative change in length between points and not the actual distance between points. This position equation describes changes in position without reference to total rope length.

$$L_{const} = 2\, s_{A/Bo} + s_{C/Bo} = 2\, s_{A/Bf} + s_{C/Bf} \tag{4}$$

$$0 = \left(2\, s_{A/Bf} - 2\, s_{A/Bo}\right) + \left(s_{C/Bf} - s_{C/Bo}\right) \tag{5}$$

$$0 = 2\, \Delta s_{A/B} + \Delta s_{C/B} \tag{6}$$

Equation 7 shows the first derivative with respect to time of both sides of Equation 3. The derivative of a constant is zero, and the derivative of position is velocity, so Equation 7 can be simplified to Equation 8. Equation 8 cleanly describes the relative and dependent velocities between the moving points.

$$\frac{d}{dt} L_{const} = 2 \frac{d}{dt} s_{A/B} + \frac{d}{dt} s_{C/B} \tag{7}$$

$$0 = 2\, v_{A/B} + v_{C/B} \tag{8}$$

The derivative of both sides of Equation 8 with respect to time again (see Equation 9) leads to the description of the relative and dependent acceleration between the points shown in Equation 10.

$$\frac{d}{dt} 0 = 2\, \frac{d}{dt} v_{A/B} + \frac{d}{dt} v_{C/B} \tag{9}$$

$$0 = 2\, a_{A/B} + a_{C/B} \tag{10}$$

Finally, the relative motion between moving points can be described in terms of the absolute kinematic properties for each point. See Equations 11-16.

$$s_{A/B} = s_A - s_B \tag{11}$$

$$v_{A/B} = v_A - v_B \tag{13}$$

$$a_{A/B} = a_A - a_B \tag{15}$$

$$s_{C/B} = s_C - s_B \tag{12}$$

$$v_{C/B} = v_C - v_B \tag{14}$$

$$a_{C/B} = a_C - a_B \tag{16}$$

Finally, for individual moving points with known kinematic properties, the absolute single particle kinematic equations can be written for each point, relating the position, velocity, and acceleration of one point at a time.

This approach can be generalized into the following steps:

1. Draw a position diagram, labeling the length of each rope segment in the pulley system.
2. Develop dependent motion kinematic relationship equations:
   - 2.1. Write the position equation by summing the constant rope segments and variable rope segments as shown in Equation 17.

$$L_{const} = L_{rope} - \sum L_i = \sum s_{is} \tag{17}$$

   - 2.2. Write the change in position equation by setting the variable rope segment side of Equation 17 at one instant equal to the variable rope segment side of the Equation 17 at another instant (Equation 18)

$$\left(\sum s_{is}\right)_o = \left(\sum s_{is}\right)_f \tag{18}$$

   - 2.3. Write the velocity equation (Equation 19) by taking the derivative of the position equation (Equation 17) with respect to time.

$$0 = \sum v_{is} \tag{19}$$

   - 2.4. Write the acceleration equation (Equation 20) by taking the derivative of the velocity equation (Equation 19) with respect to time.

$$0 = \sum a_{is} \tag{20}$$

   Note: Remember to take the derivatives of the length of each rope segment. Some segments may be of the same length; therefore, like terms may be collected to simplify the equations.

   Note: Check the reasonableness of each equation against the initial position diagram.

   Note: The derivatives with respect to time represent the change in variable segment lengths, also known as the relative velocities and relative accelerations between points. These velocities and accelerations may or may not be equal to the absolute velocity and absolution acceleration of any particular moving point in the pulley system. (See step 3).

3. Develop absolute particle kinematic relationship equations for moving points with known properties.

4. Develop relative motion kinematic relationship equations for points where variable rope segments connect two points.

   Note: For the example case above, because point B is fixed, the velocity, v_B, and acceleration, a_B, of point B is zero. Therefore, the relative motion equations indicate that the relative kinematic properties for moving points A and C are equal to their absolute kinematic properties.

This theoretical approach is sound and requires careful thinking from first principles but fundamentally takes a physicist's approach to the problem. A strong mental model is always required for engineering mechanics [9], [10], but the interaction between a complex reality and a first-principles procedure leading to dependent, absolute, and relative motion systems of equations may be more involved than necessary.

Traditional Presentation

A common simplification of the theoretical derivation removes the need to develop relative motion equations by referencing the absolute position of each moving point from a convenient fixed origin. Again, the total rope and the rope segments around the pulleys are of constant length. The sum of the variable rope segment lengths results in a position equation that describes the position of each point compared to a constant. However, in this case, the position equation is framed in terms of the absolute position of each moving point rather than the relative position and segment lengths between points. Finally, the calculated constant merely approximates the total rope length and only conceptually addresses the constant rope segment lengths. The change in position can be described by setting the initial position equal to the final position, and first and second derivatives of the position equation describe the dependent distribution of velocities and accelerations in the pulley system.

This approach is more generalizable but does not establish generalizable equations.

1. Draw a position diagram. Establish the coordinates for each moving point from a convenient common reference point (the origin of the coordinate system).

2. Develop dependent motion equations:

   2.1 Write a position equation describing the approximate total rope length using the moving point positions and geometric constraints from the diagram.

   2.2 Write the change in position equation by setting the position equation equal to itself at two different instances

   2.3 Write the velocity equation by taking the derivative of the position equation with respect to time.

   2.4 Write the acceleration equation by taking the derivative of the velocity equation with respect to time.

3 Develop absolute particle kinematic relationship equations for moving points with known properties.

Consider steps 1 and 2 of this procedure applied to the same simple case:

Step 1

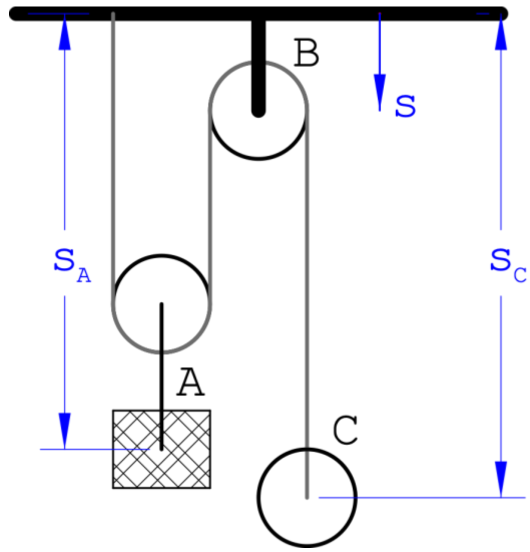


Figure 2. Kinematic property diagram showing absolute position of each moving point in the system.

Step 2.1 $$L_{const} = 2\ s_A + s_C \tag{21}$$

Step 2.2 $$2\ s_{Ao} + s_{Co} = 2\ s_{Af} + s_{Cf} \tag{22}$$

Step 2.2 $$0 = 2\ \Delta s_A + \Delta s_C \tag{23}$$

Step 2.3 $$0 = 2\ v_A + v_C \tag{24}$$

Step 2.4 $$0 = 2\ a_A + a_C \tag{25}$$

Developing these equations requires a mental model that is not overly concerned with the various constants lengths that contributes to the definition of the constant, L_{const}. The constant may explicitly incorporate constant lengths from fixed positions, arc lengths and pulley-to-object connections, but these constants disappear in the equations for change in position, velocity, and acceleration. Students and engineers can specify these lengths, to the degree they desire, but this conceptual fuzziness can create uncertainty for students.

Novel Procedural Approach

The novel approach presented in this CRI provides simplified guidelines to develop equations identical to the traditional approach. One significant limitation must be noted: the approach below is only appropriate for one-dimensional pulley problems. In other words, the direction of motion must be parallel to the ropes.

1. Draw a kinematic property diagram:

   1.1. Indicate the positive sign convention for the change in position (Δs_i) for each moving point in the pulley system.

   Consider *away* from the closest fixed point as *positive*.

   1.2. Calculate the rope coefficient (n_i) for each moving point.

   Count the number of times the rope interacts with each moving point.

   Determine the coefficient sign by considering the rope when the point is moved in the positive direction (away from the nearest fixed point):

   - If the rope goes into *tension*, the rope coefficient is *positive*.
   - If the rope goes *slack*, the rope coefficient is *negative*.

   An alternative mental model determines the coefficient sign by considering the motion of the point when it pulls the rope. The sign of the coefficient matches the direction of motion that pulls the rope, keeping it in tension:

   - If the point moves in the *positive* direction (away from the closest fixed point) and keeps the rope in tension, the coefficient is *positive*.
   - If the point moves in the *negative* direction (toward the closest fixed point) and keeps the rope in tension, the coefficient is *negative*.

   See the examples in Figure 4 and Figure 6 for additional clarity.

2. Write the dependent motion kinematic relationship equations using the rope coefficients:

   2.1
   $$0 = \sum n_i \, \Delta s_i \tag{26}$$

   2.2
   $$0 = \sum n_i \, v_i \tag{27}$$

   2.3
   $$0 = \sum n_i \, a_i \tag{28}$$

3. Develop absolute particle kinematic relationship equations for moving points with known properties using the sign conventions established in the original kinematic property diagram.

Consider the application of the procedural approach to the same simple system:

Step 1.1

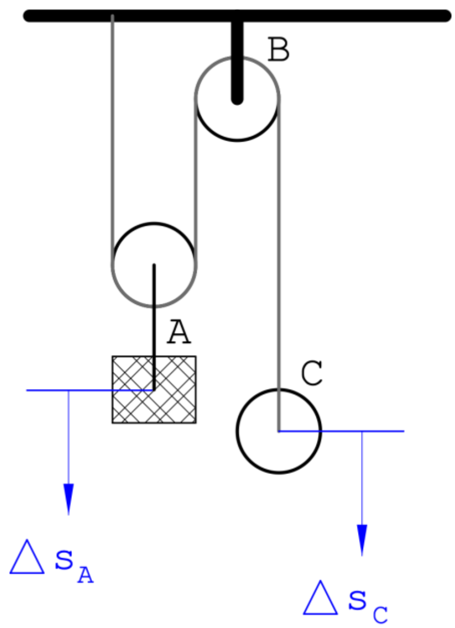


Figure 3. Kinematic property diagram showing positive sign convention for points, *A* and *C*, away from nearest fixed-point, *B*.

Step 1.2

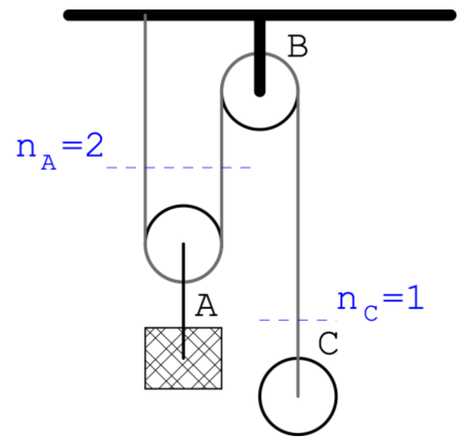


Figure 4. Rope coefficients for points *A* and *C* based on the number of interactions with rope *ABC*.

Step 2.1

$$0 = 2\,\Delta s_A + 1\,\Delta s_C \quad (29)$$

Step 2.2

$$0 = 2\,v_A + 1\,v_C \quad (30)$$

Step 2.3

$$0 = 2\,a_A + 1\,a_C \quad (31)$$

The representative constant, L_{const}, and absolute position are not typically required to solve pulley kinematic problems. If absolute position is desired, a global coordinate system and initial positions can be added to the change in position results.

The procedural approach really shines when considering a more complex, multi-rope system. The rope connecting points *A*, *B* and *C* is the same as the simple example above. Consider the procedure applied to the rope connecting points *D*, *C*, and *E* as seen in Figure 5 and Figure 6.

Step 1.1

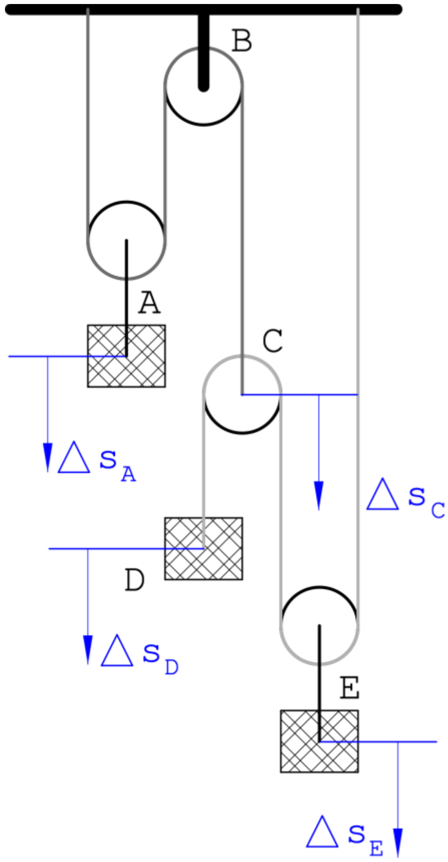


Figure 5. Kinematic property diagram showing positive sign convention for points *A, C, D,* and *E* away from nearest fixed-point *B*.

Step 1.2

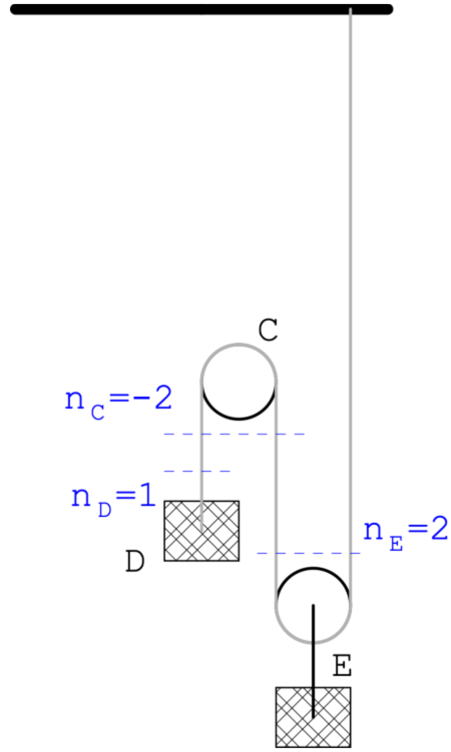


Figure 6. Rope coefficients for points *D*, *C,* and *E* based on the number of interactions with rope *DCE* and sign convention.

Step 2.1
$$0 = 1\ \Delta s_D - 2\ \Delta s_C + 2\ \Delta s_E \tag{32}$$

Step 2.2
$$0 = 1\ \Delta v_D - 2\ v_C + 2\ v_E \tag{33}$$

Step 2.3
$$0 = 1\ a_D - 2\ a_C + 2\ a_E \tag{34}$$

In this case the rope coefficient for point *C* is negative because when point C moves and pulls on the rope, it moves up in the negative direction; therefore, the rope coefficient is negative. Alternatively, when point *C* moves in the positive direction (down, away from the nearest fixed point, *B*), the rope connecting *C* to *D* and *E* goes slack, requiring a negative sign. Equations 32-34 can be solved simultaneously with Equations 29-31 and the absolute kinematic relationship equations that might be written as part of Step 3 in the procedure.

Limitations:

As noted above, the novel procedural approach is only appropriate for one-dimensional pulley problems. If there is an angle between the direction of motion of a point and attached rope(s), the traditional or first principles approach must be used. An example that violates this principle would be a pulley system in which the free end of the rope is attached to a point moving along a surface (consider a worker hauling a load by holding the free end of the rope and backing up instead of just pulling). In this case, the equation describing the length of the rope will be non-linear due to the changing angle of the rope relative to the motion of the point. However, the novel procedural approach above serves well for common applications in similar fashion to the use of gear ratios to describe rigid body dependent motion.

Classroom Ready Innovation

Learning Objectives

The following lecture outline and physical models support mastery of the following learning objectives.

- For a system involving pulleys…
  - Draw the kinematic property diagram (KPD).
  - Write the dependent motion (DM) kinematic relationship equations (KRE) from the KPD.
  - Calculate the kinematic properties (KP) for moving points.

Classroom Activity

Before coming to class, students are required to read the dependent motion section of a standard dynamics textbook [6]. In class, the discussion starts with simple questions that describe in words the behavior of a simple pulley system (See Appendix A for questions administered using Plickers [11]). The professor then demonstrates the behavior using a pulley system attached to the white board constructed from magnetic hooks, parachute cord, and metal pulleys as seen in Figure 7(a).

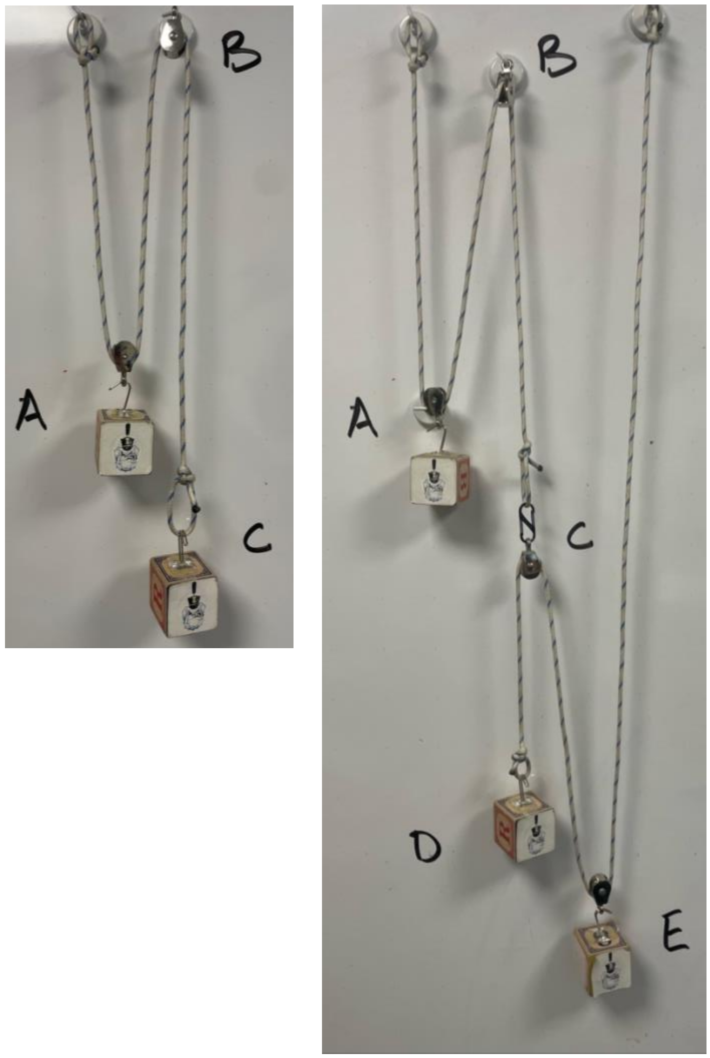
The instructor then introduces the simplified procedural approach, abbreviated for the white board as shown below:

1. Establish sign convention, Δs_i
   (away from fixed is positive)

2. Count ropes, n_i
   (pull the rope:
   sign matches motion)

3. Write equations
   (Equations 24-26)

Working together, the class develops the equations for the simple (Figure 3) and complex systems (Figure 5) and checks them against physical models (Figure 7) for the same systems. With the theory and procedure established, students work in groups on a larger numerical example from the textbook as remaining class time allows.



(a) (b)

Figure 7. Physical models of (a) a simple pulley system and (b) a complex pulley system

Conclusion

The procedural approach to developing dependent equations for pulley systems is straight forward, comprehensive and easy to remember for many students. The author experience working with colleagues teaching dynamics over the last decade suggests this approach is an improvement over the approaches to one-dimensional pulley kinematics as documented in many current textbooks.

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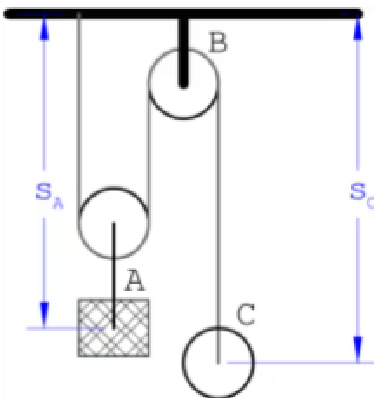
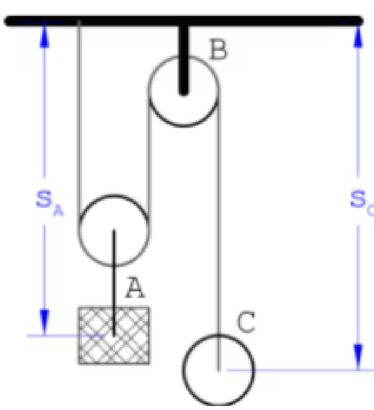
Appendix A. Pulley In-class Plickers Quiz

1 If C moves down, point A will move...



- A down
- ✓ up
- C left
- D right

2 If away from B is positive and C moves down, point A will move...



- A down and negative
- B up and positive
- C down and positive
- ✓ up and negative