CRI: Area Moment of Inertia with 3D-Printed Spinning Models

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Abstract

Area moment of inertia, though applied in solid mechanics courses, is typically taught in engineering statics courses. Students have some familiarity with mass moment of inertia from physics, but the mathematical abstraction into area moment of inertia with its odd units (length to the fourth), yet unclear application (stress analysis on beam cross-sections), marginal relation to other course content (moments and centroids), and potential notational ambiguities (in the parallel axis theorem) result in confusion and a lack of a meaningful mental models. This classroom ready innovation (CRI) extends previous published classroom activities by relating area moment of inertia back to mass moment of inertia and demonstrating the relative scale of moment of inertia through spinning 3D-printed models. Context generating examples coupled with scaled models of analyzed cross-sections encourages student engagement and mental model development. Pre- and post-lecture concept quizzes motivate and assess student understanding as related to future design and analysis choices.

Keywords

moment of inertia, statics, classroom ready innovation (CRI)

Introduction and Literature Review

Statics provides foundational content for many engineering disciplines and courses. Some statics concepts lend themselves to clear mental models and obvious applicability, *e.g.* solving for reactions. Other topics tend to be more mathematical with applications that will only be clear in following courses. Area moment of inertia is one such critical but abstract concept. Though it governs beam behavior in solid mechanics and related design fields, its introduction in statics is frequently highly mathematical and largely disconnected from other topics in statics [1], [2]. At the same time, the engineering education literature has long established the usefulness of physical models, demonstrations, and aids in teaching new concepts [3], [4], [5], [6], [7], [8]. A clear demonstration or hands on experience can helping students build mental models that accurately reflect the real world and equip the student to evaluate their analytical work [9], [10]. This classroom ready innovation (CRI) expands on previous physical modeling used to teach centroids [11] as well as moment of inertia, product of inertia, and principle axes [12]. These classroom activities rely on the proportional relationship between area moment of inertia (the new concept) and mass moment of inertia (a concept frequently taught in introductory physics courses). A set of 3D printed spinning models helps students build mental models of the relative magnitude of moment of inertia about particular axes.

Learning Objectives

The following lecture outlines and physical models support mastery of the following learning objectives.

- Calculate the moment of inertia of areas about various axes.
  - Moment of Inertia
    - Describe…
      - Area moment of inertia.
      - Area radius of gyration.
      - Polar moment of inertia.
      - Mass moment of inertia.
      - Mass radius of gyration.
      - Parallel Axis Theorem.
      - Product of inertia.
    - Calculate area moment of inertia for…
      - Continuous 2D shapes using integration.
      - Composite bodies using the Parallel Axis Theorem.
  - Describe product moment of inertia for…
    - Continuous 2D shapes.
    - Composite bodies using the Parallel Axis Theorem.
  - Describe principal moments of inertia using…
    - Analytical equations.
    - Mohr's Circle for moment of inertia.

Physical Models and Scope

The model at the center of this CRI is a frame capable of supporting a variety of thin, 3D printed shapes such that they can be spun about a particular axis as seen in Figure 1.

Introduction to Area Moment of Inertia

A mathematical introduction to area moment of inertia that explores the development of moment about an axis from a linearly increasing pressure has been illustrated in greater detail elsewhere [12]. Linearly increasing pressures are common in small deformation beam mechanics (solid mechanics) and submerged plate hydrostatics (fluid mechanics), making area moment of inertia exceptionally useful in those future courses. This presentation of the area moment of inertia connects well to the previously taught topics of distributed loads and equivalent moments, but the relative magnitudes of the new area moment of inertia property with units of length to the fourth remains abstract.

As "the unknown must be explained by means of the known", the relative magnitudes of area moment of inertia may be helpfully compared to mass moment of inertia as used for rotating object (dynamics) [13]. The thin rectangular shapes (Figure 1.r and Figure 1.s) are used to model the classic spinning figure skater illustration of mass moment of inertia. The mass moment of inertia is a critical aspect in dynamics, where larger mass moments of inertia indicate a greater resistance to change in angular velocity. The two masses at a distance (Figure 1.e) demonstrate the physical meaning of radius of gyration.

For thin shapes of constant thickness and density, it can be shown that the mass moment of inertia is proportional the area moment of inertia for a given shape, *i.e.* the mass moment of inertia for a thin plate may be estimated by multiplying the area moment of inertia by the thickness and density. Thin shapes of differing area moment of inertia (a quantity useful for beam bending and submerged sluice gate design) will have proportionally differing mass moments of inertia (easily illustrated by spinning the shape about a given axis). This is the fundamental usefulness of the thin 3D printed shapes in Figure 1 for teaching area moment of inertia. Though area moment of inertia and mass moment of inertia have very different use cases and physical meanings, they are proportional to one another: a shape with large or small area moment of inertia will also have large or small mass moment of inertia.

Calculation by Integration and Standard Shapes

The numerical example problem from the first class period involves using integration to calculate the moment of inertia about the centroidal axes for a rectangle. Having symbolically calculated the moment of inertia equation, students are introduced to the tables of moment of inertia equations in the FE Reference Handbook [14] and textbook. The rectangles in Figure 1.f and Figure 1.g illustrated the relative magnitude of the mass moment of inertia (and by proportionality the area moment of inertia) depending on the reference centroidal axis.

Parallel Axis Theorem

In the second class period, the parallel axis theorem is introduced by comparing shapes Figure 1.h and Figure 1.i to Figure 1.f and Figure 1.g and their relative mass moments of inertia. The moment of inertia (both mass and area) is larger for axes not passing through the centroid, therefore an additional second moment (of area or mass respectively) must be added to the centroidal moment of inertia.

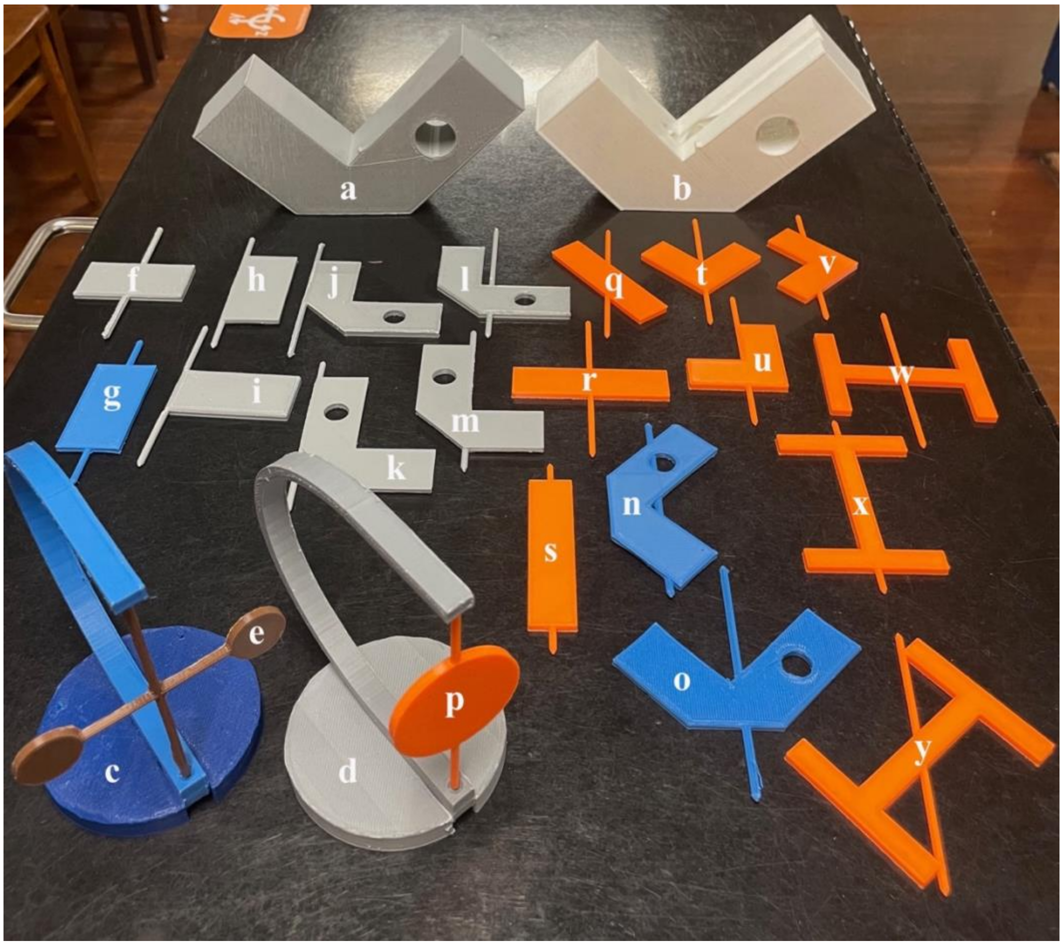


Figure 1. 3D printed models including large models (a, b) in the back from previous papers [11], [12], supporting frames in the bottom left corner (c, d), and a range of thin shapes spinnable about particular axes (e through y).

Composite Body Theory

Students are then confronted with the challenge of calculating the area moment of inertia for the arbitrary shapes Figure 1.a and Figure 1.b in the context of eventually calculating the mass moment of inertia as part of designing the thin floor and roof of a Star Tours like amusement park ride. Students use the composite body theory expansions of the parallel axis theorem to calculate the area moment of inertia for the shape around its edge axes (proportional mass moment of inertia illustrated with Figure 1.j and Figure 1.k) and then use the parallel axis theorem again to calculate the composite body area moment of inertia about the centroidal axes (again proportional mass moment of inertia illustrated with Figure 1.l and Figure 1.m). Spinning the parallel axis shapes helps students understand why they must *subtract* the second moment portion from the edge axis moment of inertia to find the minimum centroidal moment of inertia. Mathematical details have been provided elsewhere [12].

Moment of Inertia and Beam Design

In a third class period, students practice calculating area moment of inertia for beam cross-sections in preparation for solid mechanics in the next semester. Students are introduced the representative strength and service limit state equations that illustrate the future use of internal forces and moment of inertia from statics in solid mechanics. Additionally, students discuss qualitative design constraints including economic, societal, environmental, and constructability concerns. In the in-class example, students calculate the area, area moment of inertia, and radius of gyration for each shape shown in Figure 2. Each of these shapes have the same area moment of inertia resulting in the same stiffness (service limit state). The two outside shapes have the same bending moment capacity (strength limit state), with a slight reduction in bending capacity for the center shape due to the greater height and distance to the extreme fiber (centroid). The other design constraints are largely dependent on material constructability and associated economic, societal, and environmental costs, e.g. while the I-shapes might be best for steel, the rectangle is constructable in concrete, and an engineered lumber I-beam might have far lower environmental and economic cost compared to a solid wood beam.

By way of additional comparison, the models in Figure 1.p through Figure 1.y each have the same cross-sectional areas with drastically different area moment of inertia and mass moment of inertia. These shapes are the same shapes used in the assessment quiz (Appendix A). Comparisons of mass moment of inertia in class are selective and not comprehensive, leaving students to do the critical thinking about the relative moment of inertia while taking the quizzes.

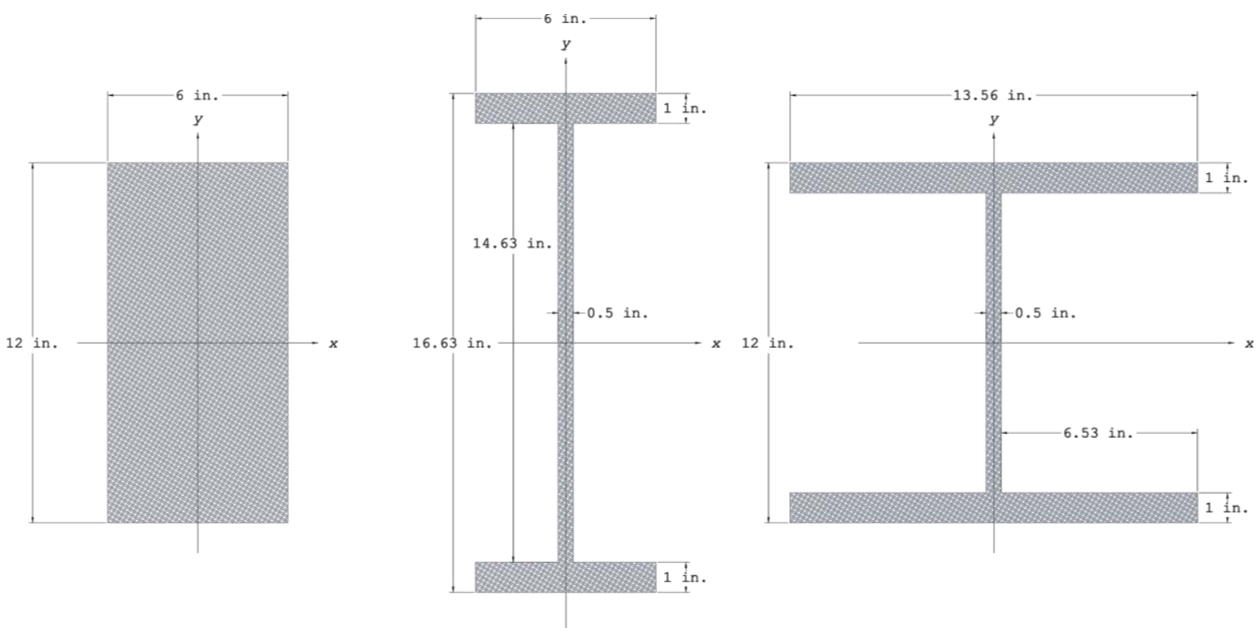


Figure 2. Example problem beam cross-sections

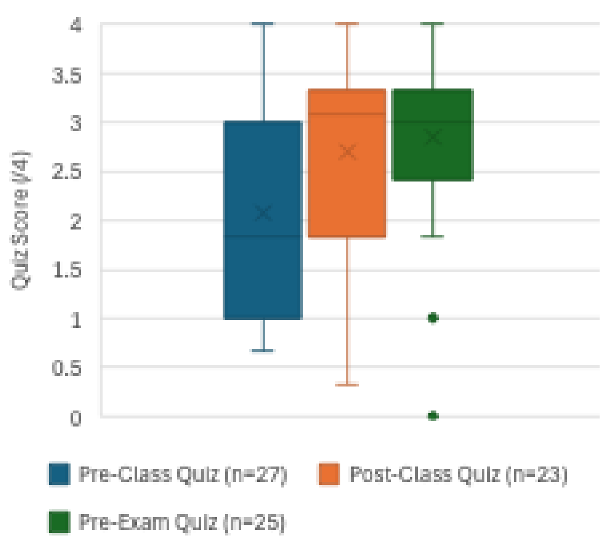
Product of Inertia and Principal Axes

In a final lecture (the last in course), students discuss the concept of product of inertia with its parallel axis theorem and composite body equations. The classroom works together to calculate the centroidal area product of inertia for the arbitrary shape (Figure 1.a and Figure 1.b). The instructor then develops the area moment of inertia Mohr's circle and identifies the principle axes [12]. The spinning models (Figure 1.n and Figure 1.o) illustrate the principal axes for the complex shape with the proportional minimum and maximum mass moments of inertia.

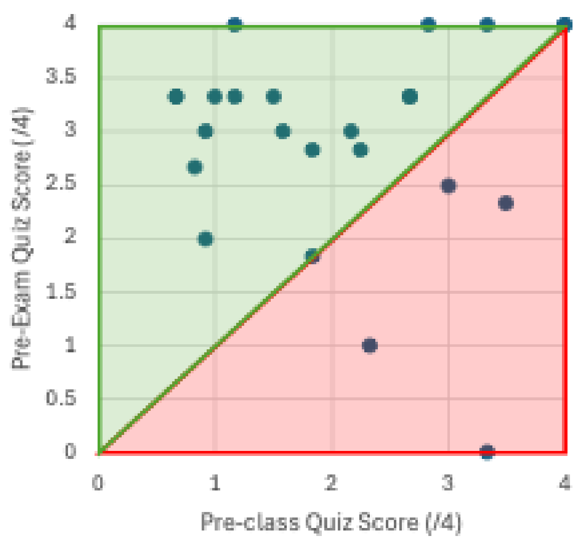
Student Assessment

Students took an identical, qualitative quiz three times during the semester (see Appendix A). The quiz was administered via the course learning management system (Canvas) and asked the students to rank order the moment of inertia for a set of shapes with identical total area. Students completed a pre-class quiz prior to the first class period on moment of inertia leveraging only on their reading and note taking. Between the second and third class periods, students repeated the quiz. The students took the quiz a final time after the fourth class period. In each quiz attempt, student received on their quiz score, not the correct answers to the quiz.

Quiz results can be seen in Figure 3. Figure 3.a illustrates how, overall, the quiz results improved with each exposure to the course content. The average grade increased from 2.1 to 2.7 to 2.9 across the three attempts. The distribution also tightens with each attempt. Figure 3.b compares initial and final quiz attempts for the 24 students who completed both assignments. The class experiences contributed to an increased quiz grade for 71% of the students, with an additional 13% seeing no change in their quiz performance. Improvements in quiz results indicate improving mental models concerning the relative magnitudes of moment of inertia.



(a)



(b)

Figure 3. Student assessment results from three (3) attempts at an identical quiz:
a) distribution of quiz results and
b) comparing first pre-class quiz attempt to final pre-exam quiz attempt (n=24).

Discussion, Application, Further Work

The 3D printed spinning models seen in Figure 1 are helpful hands-on demonstrations to illustrate the relative magnitude of moment of inertia in a statics course. These models are easy to print on any 3D printer. While this application used a single model set, manipulated by the instructor in a class of less than 30, in larger class sizes, additional copies of the models could be printed and distributed to smaller groups of students. Where statics courses are supplemented by a recitation or lab period, these models could be easily implemented as a hands-on experiment in which students confirm their quiz answers. Alternatively, students could prepare moment of inertia Mohr's circles to evaluate the principal axes for each of the shapes with identical areas. Regardless of the implementation, the ease associated with creating these 3D printed models opens new opportunities for exploration and learning.

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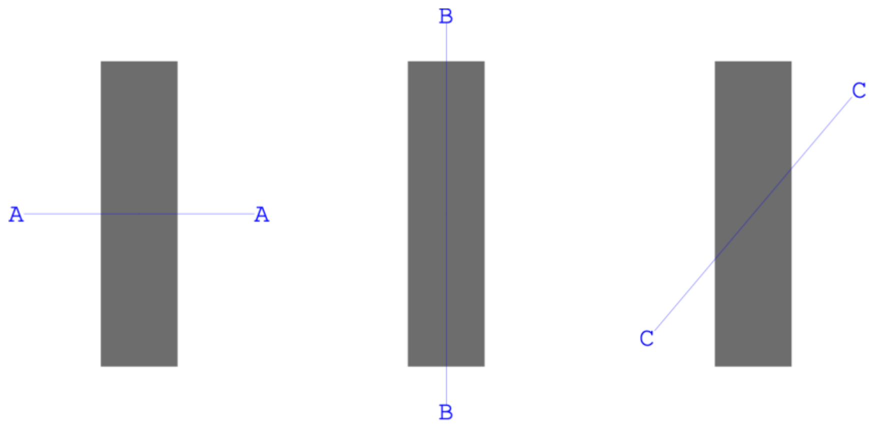
Appendix A. Moment of Inertia Quiz

For each set, please order the moment of inertia about each axis from least to greatest:

1



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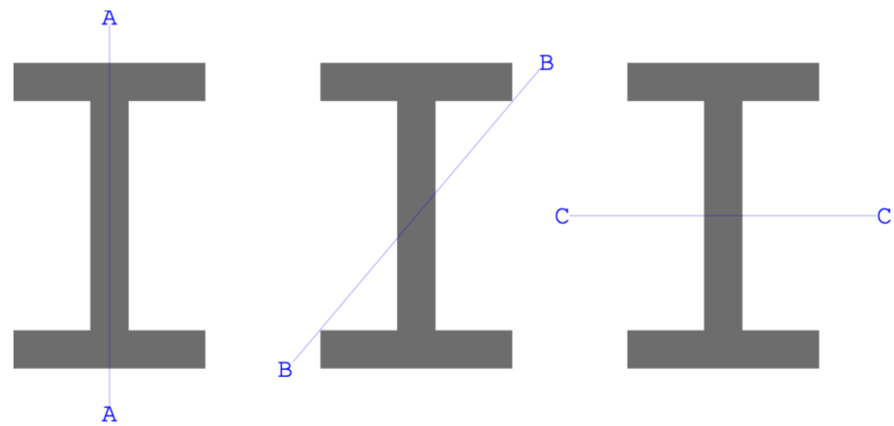

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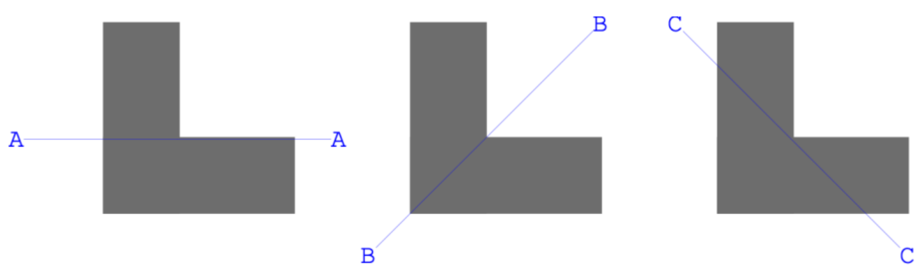

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