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Integrating Linear Algebra into Differential Equations: A Systems-First Approach for Engineering Students

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Abstract

We propose and outline here a restructuring of the traditional undergraduate first course in Differential Equations. With engineering students as its target audience, we place early and targeted emphasis on linear algebra concepts in context and in service of a systems-first approach. Unlike traditional treatments, here all higher-order ordinary differential equations are systematically reduced to first-order systems, enabling students to approach problems uniformly, regardless of order. Linear algebra—including eigenvalues, eigenvectors, fundamental sets, diagonalization, and matrix exponentiation—is introduced and used repeatedly as an essential analytical tool for solving constant coefficient linear systems, which typically account for at least two-thirds of the course content. Unifying our treatment of linear constant coefficient systems is the fundamental matrix $e^{\mathbf{A}t}$. In addition to not requiring that the coefficient matrix be diagonalizable, $e^{\mathbf{A}t}$ links integrating factor, variation of parameters, and convolution in ways that are meaningful to engineering applications. Nonlinearity is shown to be sometimes treatable with Calculus, but in general requires linearization techniques to return to the ability to apply linear algebraic tools. This integrated approach enhances students' ability to analyze coupled systems, assess stability, and apply techniques to mechanical vibrations, electrical circuits, and emerging applications in data-driven engineering. Early assessment indicates improved conceptual understanding, stronger connections between mathematical theory and engineering practice, and greater confidence in applying linear algebra in dynamic system analysis. This work contributes new practices to the field of engineering education by demonstrating that a unified, systems-oriented approach provides a more coherent and application-ready foundation for engineering undergraduates.

The central claim of this work is that the matrix exponential $e^{\mathbf{A}t}$ provides a unifying analytical framework that can replace the traditional "method-per-topic" structure of introductory differential equations.

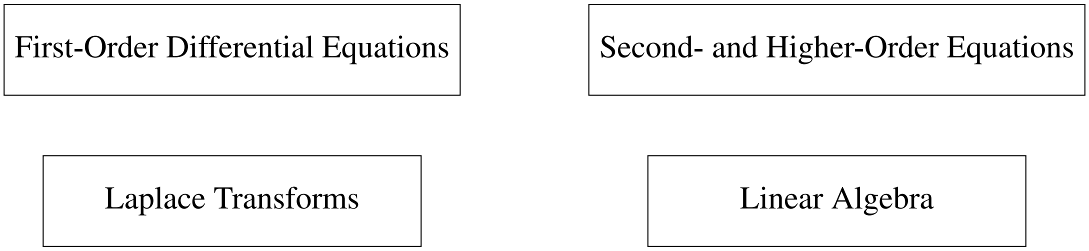

Figure 1: Traditional modular course structure: four largely independent modules.

Figure 1 depicts the common modular organization of the topics in in a traditional course: (1) first-order differential equations (separable, exact, linear, Bernoulli), (2) second- and higher-order equations (typically constant coefficients, with occasional inclusion of Euler–Cauchy forms), (3) Laplace transforms, and (4) linear algebra. This modular perspective is reflected in several widely used textbooks that explicitly note the flexibility of chapter ordering and suggest that topics may be taught in a variety of sequences [1, 2, 3, 4, 5]. While this flexibility is pedagogically convenient, it also signals the absence of a single unifying theme that consistently links systems, solution methods, and applications across the course.

Both authors of this paper have taught courses following this structure and have experimented with reordering these components. Initially our experiments sought simply to answer questions like: (i) how do we make the linear algebra responsive to and integrated with the efforts to understand, classify, and solve the respective differential equation(s)? (ii) is the bag of tricks (one for each type of ODE) really the best we can do? Is there some way(s) to unify these seemingly disparate approaches? We later consolidated answers to those questions with justification for a systems-first approach and a prioritization of matrix exponentiation (for constant coefficient systems).

Somewhat surprisingly, a clear unifying theme emerged. Rather than treating differential equations and linear algebra as adjacent topics, the four modules can be organized around a single analytical object: the matrix exponential, $e^{\mathbf{A}t}$. As we show below, this object serves as a unifying framework in which linear systems, solution methods, reductions, and applications are organically, intuitively, and tightly linked.

Given an $n \times n$ system ($n \geq 1$) of constant-coefficient first-order linear ordinary differential equations, or an individual linear constant-coefficient equation of order $n > 1$, we move seamlessly from variation of parameters—reducing to the integrating factor in the scalar case in the t-domain—to its Laplace-domain analogue involving the resolvent $(sI - A)^{-1}$ and convolution with the forcing function and the product of the transfer function and the forcing function in the s-domain (convolution).If $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$, then

$$e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)\,ds = \mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s)\right\} = e^{\mathbf{A}t}\mathbf{x}_0 + \left(e^{\mathbf{A}t} \star \mathbf{g}(t)\right)$$

.

We therefore propose a course structure that begins with systems of linear, constant-coefficient equations, first homogeneous and then nonhomogeneous, and consistently leverages the matrix exponential as the primary mathematical device toward a solution. Importantly, this reorganization does not increase the technical difficulty of the course; it remains grounded in standard calculus and algebra prerequisites and is paced appropriately for a single semester.

Systems

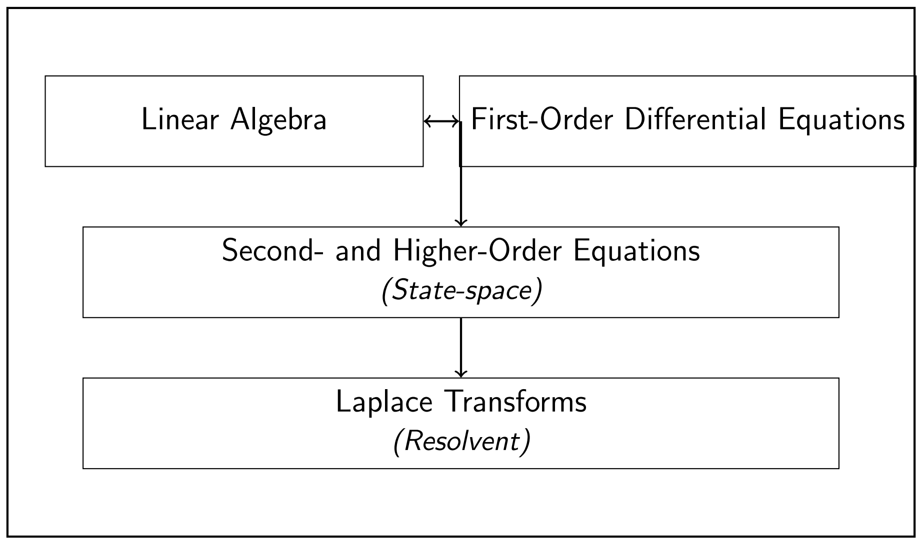

Figure 2: Unified Systems view of linear algebra and differential equations.

Representing higher order linear equations as systems of first order linear equations.

Systems of, rather than individual, equations mirror reality and the true contexts in which engineering students encounter ordinary differential equations. However, the traditional course, in which linear algebra is treated as an *add-on*, making it difficult to confront and respond to this reality. Further, every higher (n^{th}) order linear ODE can be represented as an $n \times n$ system of 1^{st} order linear ODEs. That we can interpret such a system as a matrix-vector equation:

- $\mathbf{x}' = \mathbf{A}\mathbf{x}$- unforced or,
- $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$- forced

$$\text{where } \mathbf{x}' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{1,1}(t) & \dots & a_{1,n}(t) \\ \vdots & & \\ a_{n,1}(t) & \dots & a_{n,n}(t) \end{bmatrix}, \text{ and } \mathbf{g}(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}.$$

gains us access to all the linear algebraic machinery (eigenvalues and eigenvectors) needed to analyze and solve such equations/systems.

For n large and $a_{i,j}(t)$ constant for all t and $i, j \in [1, n]$, matrix exponentiation is an efficient tool for computing such solutions, irrespective of the presence of repeated eigenvalues and the potential deficiency of A.

In general, the cases of nonconstant coefficient equations/systems are limited to a predictable short list of special examples, in the undergraduate curriculum. Many of these can be *reduced*, with an appropriate substitution or integration factor, to a constant coefficient analog. Roughly two-thirds of any semester-long first course in ODEs is dedicated to equations with constant coefficients. Presenting a unifying framework that is invariant under changes to dimension and renders previously separate approaches to simply special cases is pedagogically refreshing at least.

Constant Coefficient (homogeneous) Linear Systems: $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} \in \mathbf{R}^{n\times n}$

For $\mathbf{A} \in \mathbf{R}^{n\times n}$, the matrix exponential of $\mathbf{A}$ is defined as

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!}.$$

And it can be easily shown that in fact,

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}.$$

This last fact extends the parallels (between e^{at} and $e^{\mathbf{A}t}$) further, so that it is not difficult to see that $\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}$ solves the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, in much the same way (but with increased dimension, $n > 1$) that $x(t) = ce^{at}$ solves $x' = ax$. Consider the elementary case where $\mathbf{A} = \mathrm{diag}(a_1, \ldots, a_n)$ so that $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponds to $\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 \\ \vdots \\ a_{n,n}x_n \end{bmatrix}$, $\mathbf{x}(\mathbf{0}) = \mathbf{x}_0 = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. The general solution to this system can be found by solving each individual equation to obtain the vector solution $\mathbf{x} = \begin{bmatrix} c_1 e^{a_{1,1}t} \\ \vdots \\ c_n e^{a_{n,1}t} \end{bmatrix}$. As the number of columns of A grows (to match the number of equations) this generalizes to a (square) matrix vector product $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x_0}$.

Eigenvalues and eigenvectors of the matrix $e^{\mathbf{A}t}$

Given any $n \times n$ system of first order, linear, constant-coefficient, ordinary differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, the coefficient matrix, A, holds all the information about the general solution of the system in its eigen decomposition. The general solution is completely determined by the eigenvalues and eigenvectors of A when A has enough linearly independent eigenvectors to form a basis for $\mathbf{R}^n$. And when A is deficient, we can use generalized eigenvectors to fill out a basis of eigenvectors. The latter scenario has traditionally been treated separately, often requiring multiple new assumptions. One additional advantage of matrix exponentiation is that it does not require that A has n linearly independent eigenvectors. In what follows, we illustrate derivations of $e^{\mathbf{A}t}$ under both conditions. In both cases $\mathbf{x}_h = e^{\mathbf{A}t}\mathbf{x}_0$.

Two cases:

- $e^{\mathbf{A}t}$ when $A = V\Lambda V^{-1}$:

In this case, $e^{\mathbf{A}t} = Ve^{\Lambda t}V^{-1}$ and the has the same n linearly independent eigenvector as A. But the eigenvalues of $e^{\mathbf{A}t}$ are $e^{\lambda t}$, for $\lambda \in \Lambda$. To see this, observe that if $\mathbf{u}$ is an eigenvector of $\mathbf{A}$ associated with eigenvalue λ, then $\mathbf{A}^n\mathbf{u} = \lambda^n\mathbf{u}$. Therefore,
$e^{\mathbf{A}t}\mathbf{u} = I\mathbf{u} + A\mathbf{u}t + \mathbf{A}^2\mathbf{u}\frac{t^2}{2} + \mathbf{A}^3\mathbf{u}\frac{t^3}{3!} + \cdots = \left(1 + \lambda t + \frac{(\lambda t)^2}{2} + \frac{(\lambda t)^3}{3!} + \dots\right)\mathbf{u} = e^{\lambda t}\mathbf{u}$. Note here that $\mathbf{x} = e^{\lambda t}\mathbf{u}$ also solves $\mathbf{x}' = \mathbf{A}\mathbf{x}$, whenever $\mathbf{u}$ is an eigenvector of A, since then $\frac{d}{dt}\mathbf{x} = \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t}\lambda\mathbf{u} = e^{\lambda t}\mathbf{A}\mathbf{u} = \mathbf{A}e^{\lambda t}\mathbf{u} = \mathbf{A}\mathbf{x}$. Once again illuminating that solutions to the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are of the form $e^{\mathbf{A}t}\mathbf{x}_0$ and are completely characterized by the eigenvalues and eigenvectors of A.

When the eigenvalues of $\mathbf{A}$ occur as a complex conjugate pair $\lambda = \alpha \pm i\beta$, Euler's formula,

$$e^{(\alpha \pm i\beta)t} = e^{\alpha t}\left(\cos\beta t \pm i\sin\beta t\right),$$

allows the matrix exponential to be written entirely in terms of real-valued sine and cosine functions. Consequently, oscillatory solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ arise naturally from the real and imaginary parts of $e^{\mathbf{A}t}\mathbf{x}_0$, without requiring complex-valued state variables.

- $e^{\mathbf{A}t}$ When A is deficient:

While eigenvalues and eigenvectors provide a complete description of the matrix exponential when A is diagonalizable, this perspective alone obscures a more general and unified approach that is both computationally efficient and pedagogically accessible for differential equations. In particular, a full treatment of canonical forms is unnecessary if the goal is to compute $e^{\mathbf{A}t}$ and thereby construct solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Instead, we leverage two fundamental results from linear algebra—the Cayley–Hamilton Theorem and Hermite interpolation—to obtain closed-form expressions for $e^{\mathbf{A}t}$ without introducing Jordan or rational canonical forms.

By the Cayley–Hamilton Theorem, every square matrix satisfies its own characteristic polynomial. If $p(\lambda)$ is the characteristic polynomial of A of degree n. Consequently, all higher powers of A can be expressed as linear combinations. This observation immediately implies that the matrix exponential from the Taylor Series expansion given previously,can be written exactly as a finite polynomial in A:

$$e^{\mathbf{A}t} = c_0(t)I + c_1(t)\mathbf{A} + c_2(t)\mathbf{A}^2 + \cdots + c_{n-1}(t)\mathbf{A}^{n-1},$$

where the scalar coefficient functions $c_k(t)$ depend only on the eigenvalues of A and their algebraic multiplicities.

Because $e^{\mathbf{A}t}$ is now expressed as a finite polynomial in $\mathbf{A}$, the Cayley–Hamilton Theorem provides a second, equally important consequence: the matrix $\mathbf{A}$ may be replaced by a scalar λ in the polynomial representation. That is,

$$e^{\lambda t} = c_0(t) + c_1(t)\lambda + c_2(t)\lambda^2 + \cdots + c_{n-1}(t)\lambda^{n-1}$$

must hold for every eigenvalue λ of $\mathbf{A}$.

Each distinct eigenvalue of $\mathbf{A}$ therefore supplies one equation relating the coefficient functions $c_k(t)$. When all eigenvalues are distinct, these equations are sufficient to uniquely determine the coefficients (and would give the same result as diagonalization). However, when an eigenvalue λ is repeated, additional equations are required.

These additional conditions arise from the assumption that the scalar identity above is continuous in λ. Since both sides represent the same function of λ, their derivatives with respect to λ must also agree at each repeated eigenvalue. Differentiating with respect to λ yields

$$te^{\lambda t} = c_1(t) + 2c_2(t)\lambda + 3c_3(t)\lambda^2 + \cdots + (n-1)c_{n-1}(t)\lambda^{n-2}.$$

Evaluating this expression at a repeated eigenvalue provides an additional equation. For eigenvalues of higher multiplicity, successive derivatives with respect to λ are matched in the same manner. This process is precisely the Hermite interpolation.

As a result, the coefficients $c_k(t)$ are determined by matching $e^{\lambda t}$ and its derivatives at each eigenvalue of $\mathbf{A}$, with the number of matched derivatives equal to the algebraic multiplicity of that eigenvalue. The appearance of polynomial factors multiplying $e^{\lambda t}$ in solutions associated with defective matrices therefore follows directly from this interpolation process, rather than from a separate theory of generalized eigenvectors or canonical forms.

The Laplace transform offers another way to obtain $e^{\mathbf{A}t}$

Given $\mathbf{x}' = \mathbf{A}\mathbf{x}$, the Laplace transform applied to both sides of this equation yields $sX(s) - \mathbf{x}_0 = AX(s)$, so $(sI - A)X(s) = \mathbf{x}_0$. Therefore $X(s) = (sI - A)^{-1}\mathbf{x}_0$, and thus $\mathbf{x}(t) = \mathscr{L}^{-1}\{(sI - A)^{-1}\}\mathbf{x}_0$. But we have shown that $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$, so indeed $e^{\mathbf{A}t} = \mathscr{L}^{-1}\{(sI - A)^{-1}\}$.

This concept is named the resolvent and is prominent in control theory. Since electromechanical systems are becoming commonplace over mechanical systems, the ability to easily work with control theory is an additional advantage of this approach to differential equations.

Nonhomogeneous Constant Coefficients lead to variation of parameters which is just convolution

The benefits of using matrix exponentiation, $e^{\mathbf{A}t}$, are perhaps most evident when applied to nonhomogeneous equivalents of systems of constant coefficient linear equations.

Consider the following system: $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$, for $\mathbf{g}(t) \neq \mathbf{0}$. Recall that $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, and we already have $\mathbf{x}_h = e^{\mathbf{A}t}\mathbf{x}_0$. Note that $\mathbf{g}(t)$ is not restricted (eliminating the need for tricks like the method of undetermined coefficients).

To see that $\mathbf{x}_p = e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{g}(s)\,ds$, we use $e^{-\mathbf{A}t}$ as an integrating factor in solving $\mathbf{x}_p' = \mathbf{A}\mathbf{x}_p + \mathbf{g}(t)$ and reverse the product rule:

$$\mathbf{x}_p' = \mathbf{A}\mathbf{x}_p + \mathbf{g}(t)$$

becomes

$$e^{-\mathbf{A}t}\mathbf{x}_p' - \mathbf{A}e^{-\mathbf{A}t}\mathbf{x}_p = e^{-\mathbf{A}t}\mathbf{g}(t)$$

and observe that

$$e^{-\mathbf{A}t}\mathbf{x}_p' - \mathbf{A}e^{-\mathbf{A}t}\mathbf{x}_p = \left(e^{-\mathbf{A}t}\mathbf{x}_p\right)'$$

thus

$$\left(e^{-\mathbf{A}t}\mathbf{x}_p\right)' = e^{-\mathbf{A}t}\mathbf{g}(t)$$

integrating both sides we get

$$e^{-\mathbf{A}t}\mathbf{x}_p = \int e^{-\mathbf{A}s}\mathbf{g}(s)\,ds$$

and since $\left(e^{-\mathbf{A}t}\right)^{-1} = e^{\mathbf{A}t}$, we arrive at

$$\mathbf{x}_p = e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{g}(s)\ ds.$$

The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ is therefore:

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{g}(s)\ ds.$$

Convolution then naturally emerges as solution method of these equations. To see this simply note that

$$e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{g}(s)\ ds = \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)\ ds = e^{\mathbf{A}t}\star\mathbf{g}(t).$$

This thus provides a natural opportunity to discuss convolution in both the t- and s-domains in the same semester of this introductory course. For if we attempt to solve the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t),\ \mathbf{x}(0) = \mathbf{x}_0$ using Laplace transforms we first obtain:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x}_0 + \mathbf{G}(s))$$

Recalling from earlier that $e^{\mathbf{A}t} = \mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}$, then

$$x(t) = \mathscr{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x}_0 + \mathbf{G}(s))\} = e^{\mathbf{A}t}\mathbf{x}_0 + \mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\cdot\mathbf{G}(s)\right\}.$$

But $\mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\cdot\mathbf{G}(s)\right\} = e^{\mathbf{A}t}\star\mathbf{g}(t)$ So indeed

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)\,ds = \mathscr{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}\mathbf{x}_0 + \mathscr{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s)\}$$

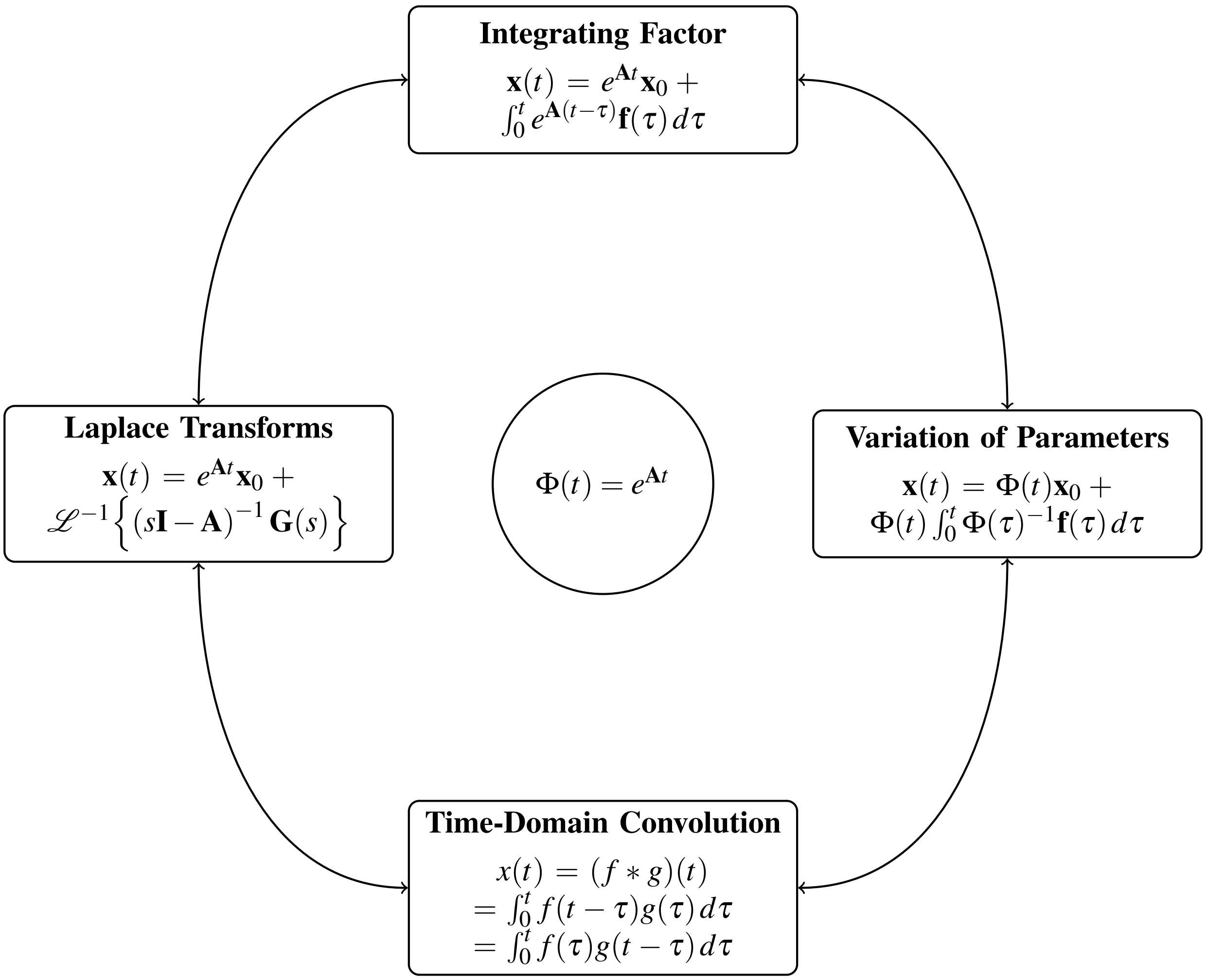

Figure 3: Equivalent methods for $e^{\mathbf{A}t}$ Nonhomogeneous solutions.

Applications

A central motivation for a systems-first approach to differential equations is that many engineering and scientific phenomena are most naturally modeled as coupled, linear, constant-coefficient systems. By introducing state-space representations and the matrix exponential early, students encounter a single analytical framework that applies broadly across disciplines rather than a collection of topic-specific techniques.

In **complex electrical circuits**, systems of differential equations arise directly from Kirchhoff's laws applied to networks containing multiple inductors and capacitors. Writing these models in state-space form allows eigenvalues to be interpreted as natural response modes of the circuit, while eigenvectors describe how voltages and currents evolve together across components. The matrix exponential provides a direct and general method for analyzing transient response and connects seamlessly to Laplace-domain techniques used in subsequent coursework.

Signals and systems provide a closely related application. Linear time-invariant systems are fundamentally governed by first-order systems, whether in continuous or discrete time. Framing convolution as a consequence of variation of parameters reinforces the idea that impulse response, system response, and frequency-domain representations stem from the same mathematical structure. The appearance of the resolvent $(sI-A)^{-1}$ as the Laplace transform of $e^{\mathbf{A}t}$ strengthens students' conceptual understanding of time–frequency duality.

In **mechanical vibrations**, multi-degree-of-freedom mass–spring–damper systems are naturally expressed as coupled first-order equations. Within this framework, eigenvalues correspond to natural frequencies and damping characteristics, while eigenvectors represent physically meaningful mode shapes. Approaching vibrations through systems rather than higher-order scalar equations prepares students to analyze coupled motion, resonance, and stability in realistic mechanical assemblies.

Applications in **seakeeping and aircraft dynamics** further illustrate the importance of systems thinking. Motions such as heave, pitch, roll, and yaw are inherently coupled and are efficiently modeled in state-space form. The systems-first approach allows students to assess stability, transient response, and the influence of parameter changes without reformulating the underlying mathematics, reinforcing intuition that extends directly to professional practice.

Beyond traditional engineering domains, the same framework appears in several **scientific applications**. In **chemical kinetics** and **reaction networks**, systems of linear differential equations describe the evolution of species concentrations near equilibrium. Eigenvalues characterize reaction time scales, while eigenvectors reveal dominant pathways and coupled behavior between species. Similarly, in **population dynamics** and **epidemiological modeling**, linearized systems near equilibrium points provide insight into growth, decay, and stability of interacting populations.

In **fluid dynamics and heat transfer**, linearized governing equations lead to system representations when discretized in space, introducing students to the connection between continuous models and large-scale computational systems. These same ideas extend naturally to **data-driven and reduced-order modeling**, where eigenvalue-based methods such as modal decomposition and system identification rely on the same underlying linear algebraic structure.

Finally, **control theory** relies almost exclusively on state-space representations and matrix methods. Concepts such as stability, feedback, and system response are grounded in eigenvalue analysis and the matrix exponential. By treating $e^{\mathbf{A}t}$ as a unifying solution concept throughout the course, students enter control systems with a coherent mathematical foundation rather than encountering a sharp conceptual transition.

Across these applications, eigenvalues and eigenvectors are not abstract mathematical objects but quantities with direct physical and scientific meaning. The systems-first approach allows multi-degree-of-freedom and multi-variable phenomena to be addressed naturally and consistently, equipping students with analytical tools that scale directly to the complexity of real-world problems.

Conclusions and Future Work

This manuscript has outlined a systems-first approach to teaching differential equations, in which the matrix exponential, $e^{\mathbf{A}t}$, serves as a unifying solution concept for both homogeneous and nonhomogeneous systems. By emphasizing reduction to first-order systems, leveraging eigenvalues and eigenvectors, and generalizing the scalar integrating factor to vector-valued variation of parameters, students gain a coherent and flexible framework applicable across engineering disciplines. Importantly, this approach is not intended to improve short-term course grades, but to build deeper intuition, conceptual understanding, and long-term ability to reason about coupled systems in subsequent coursework and professional applications.

Looking forward, this framework naturally extends to eigenfunction-based methods for series solutions of differential equations, including classical boundary-value problems and partial differential equations. By framing eigenfunctions as infinite-dimensional analogues of eigenvectors, students can apply the same unifying principles developed for finite-dimensional systems to Fourier series, Sturm–Liouville problems, and modal decomposition in PDEs. This continuity promises a pedagogically consistent bridge from ordinary to partial differential equations, enabling students to tackle more complex phenomena such as vibrations of continuous media, heat conduction, and fluid flow while retaining the structural insights gained in the linear algebraic setting.

Future work will also investigate the integration of computational tools that exploit these principles, such as numerical matrix exponentiation, modal analysis, and reduced-order modeling, to further align the course with contemporary engineering practice. Collectively, these extensions highlight the potential for a single, unified analytical perspective to span the breadth of linear differential equations, from introductory ODEs through advanced applications in scientific and engineering modeling.

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