

Work-in-Progress: Reflections on Dynamical Systems Curriculum and Pedagogy

Lauren Lazarus, Wentworth Institute of Technology

Lauren Lazarus Melfi is an Assistant Professor of Applied Mathematics in the School of Computing and Data Science at Wentworth Institute of Technology. She most commonly teaches differential equations, linear algebra, and the calculus sequence. Her research in dynamical systems focuses on coupled oscillator networks and bifurcations in delayed oscillator models.

Work-in-Progress: Reflections on Dynamical Systems Curriculum and Pedagogy

Abstract

In Spring of 2024, our institution offered a math course called “Dynamical Systems and Chaos”, which served as an elective course for engineering and applied mathematics students. The purpose was to introduce students to the analysis of long-term behaviors and stability in nonlinear dynamical systems, which have applications in modeling real-world problems. Faculty used active learning pedagogy in the classroom and made regular adjustments to center student discussion and questions. Beyond homework and exams, students completed a challenge project centered on mathematical modeling with differential equations.

Topics for this course followed a typical introductory syllabus on nonlinear dynamics, focusing on equilibrium solutions and limit cycles in systems of first-order differential equations, along with stability and bifurcation analysis. Classic examples of chaos in continuous systems and discrete maps were also explored. The course included an additional learning outcome targeted at relevant applications of dynamical systems. Since mathematical modeling is foundational to both engineering and applied mathematics, this aspect of the course could be developed further towards student interests. Student and faculty feedback on the course was generally positive but mentioned some conceptual and computational difficulties with the material.

It was important to reflect on the initial offering of this course in the context of our engineering and applied mathematics programs. Given our institution’s focus on career preparation and real-world problem solving, future offerings present an opportunity to develop students’ interest and better meet their needs. In this paper, we will give details about the course and student feedback. Possible curricular and pedagogical changes will also be discussed.

Introduction

This work-in-progress paper discusses the design and implementation of a “Dynamical Systems and Chaos” course as an upper-level undergraduate elective at Wentworth Institute of Technology. The course can serve as a technical elective for majors and minors in applied mathematics, with many students in engineering or computer science majors pursuing this minor. The course material combines topics from differential equations, linear algebra, computer programming, and more broadly, mathematical modeling of physical and social systems. As such, it provides a domain for students to synthesize their mathematical knowledge and apply it to real-world problems.

Many examples of dynamical systems are most clearly related to topics in engineering and the natural sciences. However, we expect opportunities for computer science and data science majors to gain from this course as well. Previous discussion has highlighted some of the connections between computer science and the study of dynamical systems [1]. Students may find it

interesting to experience how computational tools and scientific programming are used in mathematical modeling and related research; meanwhile, stability and other concepts from dynamical systems theory can be applied to topics in computer and data science including machine learning, neural networks, and algorithms. Over the years, general growth in a variety of programs dedicated to computational science and engineering acknowledges the importance of interdisciplinary efforts to understand and model the behaviors of complex systems [2], [3].

Graduating approximately 370 students from the School of Engineering and the School of Computing and Data Science combined in 2024, Wentworth Institute of Technology is a primarily undergraduate institution with a strong emphasis on career readiness. Students are required to complete two semesters of co-op experience, and many classes emphasize hands-on and experiential learning environments. The National Association of Colleges and Employers (NACE) identifies eight main competencies for career readiness, including critical thinking, technology, communication, and teamwork [4]. We note that several of these competencies are correlated with the exploration of open-ended questions using mathematical modeling [5]. In support of these goals, our course design includes a collaborative modeling project promoting the application of dynamical systems methods to problems derived from active scientific research.

Thorough analysis of nonlinear dynamical systems has been considered an advanced topic and sometimes relegated to graduate courses in the past. However, some work has been done designing courses to expose students to these concepts in their undergraduate study, commonly in relation to their fields of interest. We note a few examples here:

- An early course in dynamics for students with a minimal background in college-level mathematics may be achieved by focusing on discrete maps and making heavy usage of numerical simulations [6].
- A later course about chaotic behavior may build from other courses within the students' program requirements, e.g. a course for electrical engineering majors may be designed around prerequisite knowledge of electronics [7].
- Other options such as multi-semester course sequencing allow for more subject material to be taught within the class time or in the context of project-based learning [8].

Designing undergraduate-level courses on advanced topics is particularly relevant for career-oriented institutions such as Wentworth, since only a small proportion of our graduates continue to graduate school, with most opting for industry jobs instead.

Course Details

The course “Dynamical Systems and Chaos” was taught at Wentworth during the spring 2024 semester. Materials for the course were based on a standard textbook *Nonlinear Dynamics and Chaos* by Steven H. Strogatz [9]. The syllabus included discussion of equilibrium solutions, linear stability, and bifurcation analysis in one- and two-dimensional systems, the creation of limit cycles through Hopf bifurcations, and a brief exploration of chaos in continuous and discrete systems via the Lorenz system and logistic map. Some specific applications were

discussed, including population dynamics and a nonlinear pendulum; the portfolio of these examples could be developed further based on the background of the students.

Although we anticipated that many students would have experience in some combination of multivariable calculus, linear algebra, or MATLAB, depending on their program requirements, the only prerequisite was a standard course in differential equations. In contrast to some other institutions, the primary differential equations course taken by students at Wentworth does not include a discussion of equilibrium solutions or analysis of linear stability. Accordingly, our dynamical systems course took a slower approach to early content topics, revisiting direction fields and Euler's method in the context of MATLAB, along with a thorough analysis of equilibrium solutions in first-order differential equations. Without guaranteed background knowledge in linear algebra, it was also useful to extend the discussion of eigenvalues and stability in linear systems beyond the basics of the solution method seen for two-dimensional systems in the differential equations course.

Homework was assigned approximately every two weeks, with most problems adapted from the textbook. Two in-class exams assessed the determination of stability and bifurcations of equilibrium solutions, in one-dimensional and two-dimensional dynamical systems respectively. The third exam, covering discrete maps and chaos, was given as a take-home exam with allowances for external resources to be used, including MATLAB for numerical integration of systems. There was no cumulative final exam.

Modeling Discussion & Project

A modeling project was given in the second half of the course as a culminating exploration of applying dynamical systems and stability analysis to real-world questions. The project assignment refers to a selection of past problems from the SIMIODE Challenge Using Differential Equations Modeling (SCUDEM) [10]. Following the structure of the SCUDEM challenge, student groups were expected to respond to one of these problems by designing or adapting a reasonable model, exploring its long-term behaviors, and answering some predictive questions about their system.

Through the design of this project, we gave students autonomy both in choosing from a selection of problems and in how they approached the modeling process. Problems were chosen such that the students could highlight knowledge and skills acquired during the course, while not requiring a specific method to be used. A required check-in with the instructor ensured that students had chosen a topic and overcome any initial hesitations in their exploration.

The students were asked to prepare both a brief presentation in class and a longer write-up of their work. The presentation session offered time for instructor feedback and for other students to contribute ideas to the ongoing work prior to the final write-up. We note that presentation skills and technical writing were not formally addressed as course learning outcomes in the main body of the course. Accordingly, evaluation of the project deliverables relating to communication

skills was focused on the students' growth experience and providing formative feedback rather than a summative grade.

Results of Spring 2024 Pilot

The initial offering of this class in spring 2024 was taken by four students in engineering and data science majors who were also completing minors in applied mathematics. Although previous experience in MATLAB was not required to enroll in the class, most students had encountered MATLAB in prior courses, with varying degrees of confidence and complexity. Student experience with other background courses was mixed and did not affect the implementation of the course beyond the level of typical concerns about content retention.

The small number of students did lead to high variability in the day-to-day atmosphere of the class, which was run as an active lecture with discussion and problem solving. The schedule of topics was frequently adjusted to meet the needs of individual students; varying student experience with a given topic was generally addressed by using class time to review or reconstruct the concepts. This was particularly evident in cases where the students were working to understand pieces of MATLAB code presented by the instructor, so that they could adapt and use it effectively on their own. In some other cases, the adaptations were in response to exploratory questions from the students, leading to interesting results through group discussion or minor changes in content coverage to satisfy the students' curiosity.

Modeling Project

Because of the small class size, the modeling project was assigned on an individual basis rather than in groups. This choice was made to support flexibility for the students, both in avoiding scheduling barriers and in choosing problems that sparked their individual interests and ideas. However, this decision also reflected the instructor's observation that teamwork had been deprioritized in the overall experience, based on the personalities involved in the class.

Of the three students who completed the project, two chose from the list of suggested challenge problems. The problems chosen were "People Ruin Everything" from SCUDEM VII 2022 [11], exploring the impact of urbanization on animals and ecosystems, and "Submitted a Tweet, Now What?" from SCUDEM VI 2021 [12], discussing social media content moderation. The third student proposed and was permitted to study the ecological system described in a published research article [13]. We contrast them through general observations here:

- The SCUDEM problems provided useful structure and guidance for these projects, as well as ensuring a reasonable scope of effort. Student work was closely related to the models and methods discussed in class, involving numerical simulation and stability analysis, and general discussion of long-term behaviors in systems.
- On the other hand, the student who explored an external project appeared to exhibit a higher level of interest and self-motivated exploration of their dynamical system. While

the investigation was less closely supported by earlier coursework, this project displayed a different aspect of the mathematical modeling mindset, oriented toward the comparison of model behavior to observed data.

Despite the different approaches, we note that all of the projects relied heavily on MATLAB for numerical simulations or generating phase plane diagrams.

The final 75-minute class session was scheduled for presentations, allowing up to 25 minutes per student to share their progress, receive feedback and questions, and discuss further exploration. Since this session took place prior to the submission of the write-up, students were encouraged to use this as an opportunity to improve their final model, analysis, and results; questions from the instructor were directed at further development of the work. This mindset is consistent with the iterative nature of the modeling process and supports growth in multiple career competencies.

Overall Observations

Experiences in the classroom were generally seen as positive. Students were typically engaged in class despite low energy levels due to it being scheduled in an unpopular timeslot. However, the assessments and course material overall were found to be challenging, both conceptually and in specific computational skills. This difficulty had an impact on the logistics of timed exams, which were extended to support each student's full attempt at the exam problems; based on this, we would suggest converting the timed exams into take-home assessments. If desired, a small-scale oral component could be attached to the assessment to ensure student ownership of the course material.

The perceived challenge level of the course may have also affected the students' confidence in approaching the modeling project, where the pathway was not immediately clear, and the calculations could easily become more difficult than instructor-designed assignment problems. Increased in-class practice of modeling behaviors and other adaptations may be needed to overcome this aspect of the course.

Computational Choices

In general, the algebraic and numerical computations required to analyze nonlinear dynamical systems did present as a source of difficulty for this course. Some example systems may be designed such that the algebra can be handled by the students directly, which is useful for initial discussion of the concepts. Beyond these (relatively simplistic) cases, the need for computational assistance to analyze more complicated models provides both challenges and opportunities, depending on the focus of the class.

Others have previously discussed or implemented the use of pre-built programs and computer algebra systems (CAS) such as Maxima [14], Maple [15], or Mathematica [16] for undergraduate dynamical systems classes. The level of student involvement in the hands-on programming of

these systems varies in the existing work depending on the level of the class, the students' coding experience, and their interest. While these systems have generally become more versatile for all types of calculations, we note that the symbolic manipulation provided by a CAS can be particularly helpful in bifurcation analysis and other calculations in multi-dimensional systems that involve many variables and parameters.

On the other hand, software such as MATLAB is commonly used for numerical simulation of dynamical systems to determine their behavior. Along with built-in MATLAB functions such as "ode45" for numerical integration [17], convenient resources have been developed for the generation and exploration of slope fields and phase portraits [18]. Numerical methods may also be useful to approximate the solutions of algebraic or transcendental equations in the process of analyzing system behaviors and stability.

For many of our students, MATLAB also has the benefit of familiarity relative to a CAS. Since Wentworth provides faculty and student access to both MATLAB and Mathematica, both programs were used on a case-by-case basis in this dynamical systems course for in-class demonstrations. However, although students were offered supporting materials for both programs, most chose to work only in MATLAB based on their previous experience in their coursework and its relative prevalence in the engineering industry.

Conclusions

In this paper, we have discussed the design and implementation of an advanced elective course in nonlinear dynamical systems. The first offering of this course experienced typical difficulties for a new elective, including low student enrollment and the need for adaptive re-alignment of course materials. However, it shows potential for further development by refocusing on a subset of technical skills and preparing students for independent work on solving imperfect problems.

The mathematical modeling and technical skills developed in this course support growth in career readiness for students in engineering and computing-oriented programs, while the topic has disciplinary relevance to these fields through various applications. Further study should include the collection of qualitative feedback from students at multiple times after the course has ended, to measure its impact on their preparedness for future co-op and job experiences. Quantitative measures of student growth on specific technical skills would also be useful for improving the course materials and determining the next evolution of this course offering.

Challenges and Future Directions

Based on the initial offering of this course, we see two primary constraints: the development of student skills in higher-level analysis and synthesis, and the limited amount of class time available to add and develop more specific skills. To narrow the discussion and find priorities within the course material, we suggest two possible directions for future versions of the course.

A more generalized version of the course would have an increased emphasis on mathematical modeling within the context of dynamics. Removing the rigorous discussion of bifurcations would provide more time to discuss modeling approaches and additional examples depending on the students' interests. By focusing on the design of dynamical models and their interpretation, the students will be better prepared to explore open-ended projects. This approach would also prioritize a specific subset of MATLAB skills to simulate each model and perform further analysis, building up from students' previous experiences with the software.

Alternately, the specialized course could be adjusted to maintain focus on dynamical systems content at a high level, while reconsidering computational priorities. By developing software usage for algebraic and numerical purposes, students may be able to gain understanding and technical skills without sacrificing class time to do algebraic computations by hand. However, this approach would suggest additional time for students to learn enough syntax within some combination of MATLAB and/or Mathematica (or another CAS) to support both numerical and analytical results. While this is a reasonable expectation for experienced practitioners, some students may find this challenging or intimidating.

On an administrative level, the course also faces challenges in student enrollment. Through informal conversation with students, we have observed that they readily appreciate the utility of concepts explored in this dynamical systems course. The name "chaos" garners attention colloquially as well. However, the viability of the course as an elective depends on its perceived usefulness towards fulfilling degree requirements and career goals, along with difficulty in scheduling it among the students' other program requirements. Future offerings should be actively promoted in differential equations classes during the previous semester(s); by recruiting students in this context, connections will be built between their current learning and further study within the discipline, even if they choose not to take the course.

There are several benefits of a moderate class size (10-20 students), particularly in relation to the modeling project, including:

- Representation of more disciplinary backgrounds and general experiences
- Development of teamwork skills
- Students acting as "consultants" for other project groups, either informally or during the presentation feedback period
- Possible coaching of student teams outside of the course to participate in the live SCUDEM challenge.

Subject to the enrollment of a larger class of both engineering and computing students, we would also like to study the impact of intentionally creating cross-disciplinary student teams for completion of the modeling project. By setting up students to share their expertise and work together towards a shared goal, we hope to see each student grow in relation to all aspects of the necessary work.

Aside from the above proposals, we would be interested in attempting to run this course with elements of inquiry-based learning pedagogy in the future. Some resources for inquiry-based learning have been developed previously for courses in differential equations [19] and discrete dynamical systems [20]. Given an appropriate amount of prior knowledge and in combination with direct instruction for specific skills, we believe that the exploratory nature of inquiry-based learning would lead naturally and effectively to the modeling project.

Acknowledgements

Grateful acknowledgements to the School of Computing and Data Science at Wentworth Institute of Technology for supporting the initial offering of this course, and to Deirdre Donovan for initial guidance in preparing this paper.

References

- [1] D. Harel and A. Marron, “Introducing Dynamical Systems and Chaos Early in Computer Science and Software Engineering Education Can Help Advance Theory and Practice of Software Development and Computing,” in *Leveraging Applications of Formal Methods, Verification and Validation*, T. Margaria and B. Steffen, Eds., Cham: Springer International Publishing, 2021, pp. 322–334. doi: 10.1007/978-3-030-89159-6_20.
- [2] P. Turner, L. Petzold, A. Shiflet, I. Vakalis, K. Jordan, and S. St. John, “Undergraduate Computational Science and Engineering Education,” *SIAM Rev.*, vol. 53, no. 3, pp. 561–574, Jan. 2011, doi: 10.1137/07070406x.
- [3] U. Rde, K. Willcox, L. C. McInnes, and H. D. Sterck, “Research and Education in Computational Science and Engineering,” *SIAM Rev.*, vol. 60, no. 3, pp. 707–754, Jan. 2018, doi: 10.1137/16M1096840.
- [4] “What is Career Readiness?,” National Association of Colleges and Employers. Accessed: Jan. 11, 2025. [Online]. Available: <https://www.nacweb.org/career-readiness/competencies/career-readiness-defined>
- [5] S. Garfunkel and M. Montgomery, *GAIMME—guidelines for assessment & instruction in mathematical modeling education*. SIAM, 2019.
- [6] J. M. Seoane, S. Zambrano, and M. A. San Juan, “Teaching nonlinear dynamics and chaos for beginners,” *Lat.-Am. J. Phys. Educ.*, vol. 2, no. 3, p. 10, 2008.
- [7] B. Flores, H. Ochoa, and C. Pappu, “Adapting Chaos Theory for Undergraduate Electrical Engineers,” in *2022 ASEE Annual Conference & Exposition Proceedings*, Minneapolis, MN: ASEE Conferences, Aug. 2022, p. 40835. doi: 10.18260/1-2--40835.
- [8] G. Bischof *et al.*, “Exposing Undergraduate Engineering Students to Nonlinear Differential Equations Using a Practical Approach in Project-based Learning Environments,” in *2020*

ASEE Virtual Annual Conference Content Access Proceedings, Virtual On line: ASEE Conferences, Jun. 2020, p. 34652. doi: 10.18260/1-2--34652.

- [9] S. H. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. CRC press, 2018.
- [10] “QUBES - SCUDEM - SIMIODE Challenge Using Differential Equations Modeling.” Accessed: Jan. 14, 2025. [Online]. Available: <https://qubeshub.org/community/groups/scudem>
- [11] “SCUDEM VII 2022 Problems.” Accessed: Apr. 25, 2025. [Online]. Available: <https://qubeshub.org/community/groups/scudem/File:/uploads/docs/2022/SCUDEM-VII-2022-problems.pdf>
- [12] “SCUDEM VI 2021 Problems.” Accessed: Apr. 25, 2025. [Online]. Available: <https://qubeshub.org/community/groups/scudem/File:/uploads/docs/2021/SCUDEM-VI-2021-problems.pdf>
- [13] S. P. Elias, J. W. Witham, and M. L. Hunter Jr., “*Peromyscus leucopus* Abundance and Acorn Mast: Population Fluctuation Patterns over 20 Years,” *J. Mammal.*, vol. 85, no. 4, pp. 743–747, Aug. 2004, doi: 10.1644/BER-025.
- [14] J. E. Villate, “Teaching Dynamical Systems with a Computer Algebra System,” in *Proceedings of SIIE '06 - 8th International Symposium on Computers in Education*, León, Spain: Universidad de León, 2006, pp. 125–131. Accessed: Dec. 18, 2024. [Online]. Available: https://web.fe.up.pt/~villate/publications/Villate_2006_Teaching_with_Maxima.pdf
- [15] R. H. Enns and G. C. McGuire, *Nonlinear Physics with Maple for Scientists and Engineers*. Springer Science & Business Media, 2012.
- [16] N. Bellomo, L. Preziosi, and A. Romano, *Mechanics and Dynamical Systems with Mathematica®*. Springer Science & Business Media, 2012.
- [17] “Solve Nonstiff ODEs.” Accessed: Jan. 14, 2025. [Online]. Available: <https://www.mathworks.com/help/matlab/math/solve-nonstiff-odes.html>
- [18] *MathWorks-Teaching-Resources/Phase-Plane-and-Slope-Field*. (Oct. 04, 2024). MathWorks Teaching Resources. Accessed: Jan. 11, 2025. [Online]. Available: <https://github.com/MathWorks-Teaching-Resources/Phase-Plane-and-Slope-Field>
- [19] C. Ekici and C. Plyley, “Inquiry-Based Modeling of Population Dynamics With Logistic Differential and Difference Equations,” *PRIMUS*, vol. 29, no. 6, pp. 553–570, Jul. 2019, doi: 10.1080/10511970.2018.1484399.
- [20] T. LoFaro and J. Ford, *Discovering Dynamical Systems Through Experiment and Inquiry*. New York: Chapman and Hall/CRC, 2021. doi: 10.1201/9781003024132.