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Towards a Cross-National Mathematical Skills Inventory for Engineers

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Work in Progress: Towards a Cross-National Mathematical Skills Inventory for Engineers

Abstract

This Work-in-Progress (WiP) paper outlines the development of a mathematical skills inventory tailored for undergraduate engineering students. While traditional concept inventories (CIs) primarily assess students' conceptual understanding, the proposed skills inventory focuses on evaluating the practical competencies and cognitive abilities essential for mastering fundamental mathematical concepts. By adopting this skill-oriented approach, the inventory aims to offer a more accurate and comprehensive evaluation of students' knowledge, skills, and abilities (KSAs), capturing deeper levels of cognitive understanding. The broader implication of this work lies in equipping educators with actionable insights that guide targeted interventions and address skill deficits effectively.

Using the concept of functions as an exemplar, this paper lays the foundation for a competency-based framework in engineering mathematics. While the concept of functions has been extensively studied within the context of mathematics students and supported by a dedicated CI, existing instruments often lack an engineering perspective. Their limited contextual relevance renders them less suitable for use with broader engineering cohorts, where foundational mathematics is embedded within practical engineering contexts and applications. Moreover, where engineering research does exist, they tend to focus on discipline-specific concepts, such as frequency responses within control theory, equivalent circuits, and Newton's laws, often overlooking the underlying mathematical concepts that support engineering learning and problem-solving.

Threshold Concepts in Mathematics

In educational literature, threshold concepts are discipline-specific ideas that act as learning bottlenecks for students due to their troublesome and transformative nature. These concepts encompass sets of KSAs that may initially seem unfamiliar, contradictory, or inconsistent but, once understood, lead to a profound and irreversible shift in the learner's perspective of the subject [1,2]. Mastering these concepts is inherently difficult, often leaving learners in a state of liminality, a transitional phase marked by incomplete and inauthentic understanding characterized by reliance on memorization [1,2]. Overcoming these "stuck places" demands not merely the acquisition of knowledge but an ontological transformation, fundamentally reshaping one's way of thinking and being [3]. Examples of threshold concepts in mathematics include limits [1,4], complex numbers [1], mathematical proofs [3] and functions [5,6].

Functions as a Threshold Concept

Functions pose two key troublesome aspects that contribute to their nature as a threshold concept. The first is their representational complexity, requiring students to integrate and translate between various modes, including symbolic, graphical, tabular and contextual forms [7-10]. The second is the process of reification, where students transition from understanding functions operationally as processes to perceiving them structurally as objects, as described in the APOS theory [5-7].

APOS (Action, Process, Object, Schema) is a constructivist model in mathematics education that describes how students construct and comprehend mathematical concepts [11-13]. It outlines four developmental stages: ‘Action’ (where students perform explicit steps to solve problems); ‘Process’ (where actions are internalized into reversible mental operations); ‘Object’ (where processes are encapsulated as manipulable entities); and ‘Schema’ (where actions, processes, and objects are integrated into a coherent framework for problem-solving).

In the context of functions, the ‘Action’ stage involves perceiving functions as explicit rules, focusing on evaluating input-output pairs or solving equations through substitution [8,9,11,12]. At the ‘Process’ stage, students internalize these operations, viewing functions as dynamic mappings (e.g., mentally visualizing a quadratic function as a parabola without explicit calculations) and recognize properties such as domain, range, and reversibility [8]. The ‘Object’ stage is characterized by the encapsulation of processes, enabling tasks such as composing, inverting, or transforming functions through shifts and stretches [12,13]. Finally, at the ‘Schema’ stage, students synthesize these ideas into a unified framework, linking multiple representations and applying function concepts to broader contexts, such as interpreting rates of change in applied problems [11,13].

The Need for a Skills-based Inventory

Building on the theoretical grounding of threshold concepts and the APOS framework, tools such as the Function CI [7,14], the Precalculus Concept Assessment (PCA) [8], and the Function Model [9] have been developed to assess understanding of functions. Though these instruments capture critical aspects of the function concept by offering insights into students’ conceptual and procedural knowledge, they also highlight pertinent gaps the proposed skills-based inventory seeks to address.

For example, in Figure 1, item 6 of the Function CI [14], requires students to match a quadratic, cubic, and reciprocal function to their corresponding graphs, with the third option serving as a distractor for the reciprocal function. However, including multiple families of functions in a single question weakens the effectiveness of distractors. With only one quadratic and one cubic option, students can easily rely on pattern recognition or basic elimination, making it debatable whether correctly matching these graphs demonstrates a genuine conceptual understanding. Students’ existing familiarity with these basic functions (particularly at HE level) further limits the question's diagnostic value, as well as the diversity of cohorts and institutions across which the tool can be implemented. Additionally, the question in Figure 1 is highly susceptible to procedural approaches, such as substituting $x = 0$ to match the resulting y-values, given that all four graphs have unique intercepts. This assesses only basic arithmetic and reflects an ‘Action’-level conception within the APOS framework; a construct from which we aspire our students to progress. While omitting axis values may prevent such procedural shortcuts and promote reasoning to some extent, this alone does not address the broader issue of failing to definitively assess whether students attain ‘Object’-level understanding, a limitation acknowledged by the authors themselves [7].

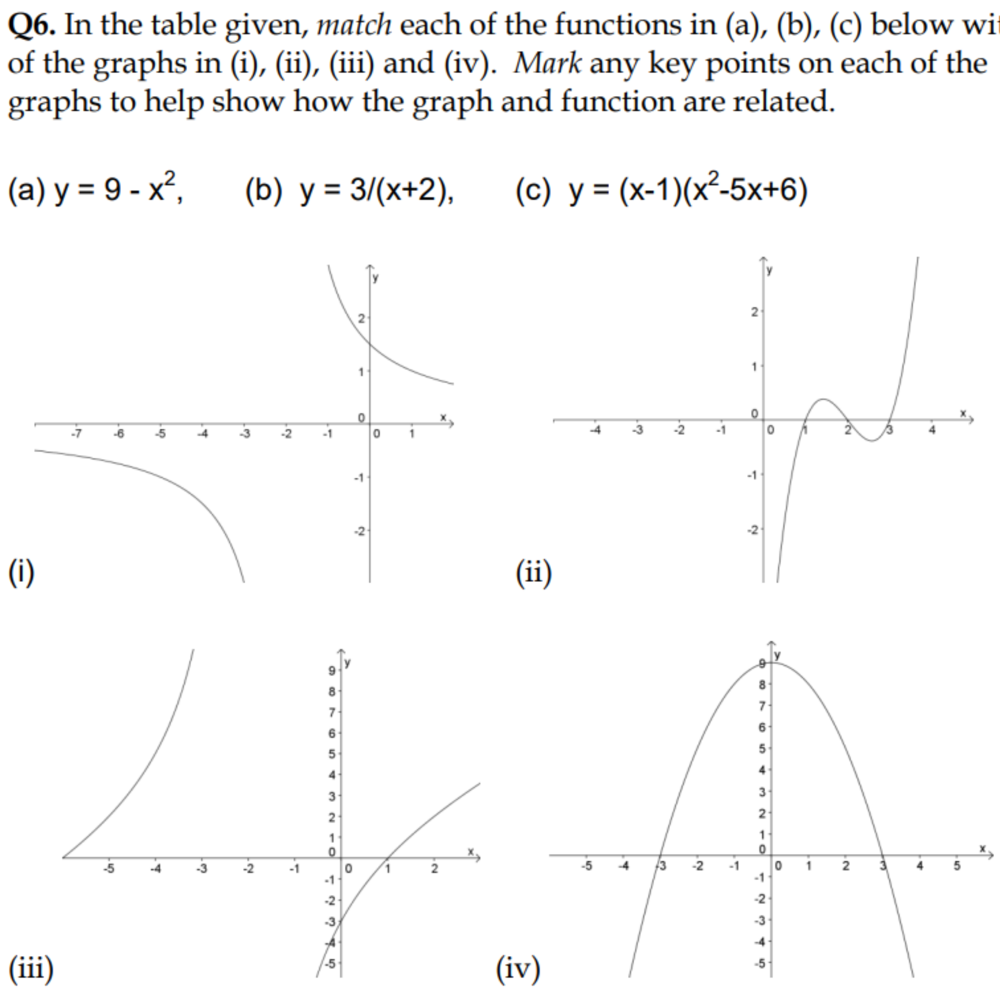

Figure 1. Item 6 from the Function Concept Inventory [14].

A more effective approach would involve restructuring the task into three separate questions, each focusing on one specific type of function. This would allow for three meaningful distractors for each question, that can test understanding more rigorously. While this introduces a trade-off by potentially lengthening the instrument and discouraging its use in classroom settings, prioritizing the ‘Object’-level handling of different function families over brevity is justified. For engineers, the ability to translate graphical representations of phenomena into equations is far more valuable than a comprehensive coverage of function types (an approach that traditional mathematical courses might emphasize more heavily).

For example, a question focused on quadratic functions could include distractors that all resemble a parabola, but differ in stretches, shifts, or reflections, compelling students to employ higher-order skills such as decomposition. In this context, students would be required to break down the function its base form and then reconstruct it through a series of transformations [9]. For example, starting with the parent function of the quadratic family, $y = x^2$, students can form a mental image of the graph shifting b units to the right to construct $(x - b)^2$, then stretch or compress this image to account for the coefficient a, and finally shift it vertically to obtain $y =$

$a(x-b)^2+c$. This process requires students to concurrently unravel the transformation coefficients while dynamically updating their mental image, thereby tapping into skills such as concurrent thinking, and multimodal translation. Such an approach fosters an understanding of relationships and transformations through advanced skills characteristic of higher-level conceptions in the APOS framework, promoting an 'Object'-like treatment of functions.

Furthermore, framing the question within an engineering context would increase its relevance and applicability to wider STEM curricula, while also assessing additional skills and competencies such as mathematical modeling and abstraction (e.g., the ability to work with simplified, abstract representations of reality). For instance, rather than asking students to identify the graphical representation of a quadratic expression, the task could be contextually designed to involve modelling the trajectory of a projectile. This shift from a theoretical exercise to a practical problem-solving scenario, such as predicting a projectile's path from a given quadratic equation, would make the question more engaging and authentic. Additionally, this approach allows customization of the type of function tested to align with instructor goals and course contexts (e.g., logarithmic functions may be more relevant than quadratics in certain engineering disciplines and contexts).

However, contextualization also introduces certain challenges. Designing questions that remain broadly accessible across a wide range of disciplines, whilst maintaining disciplinary relevance, requires careful selection of contexts that are neither too niche nor too generic. Moreover, as many contextualized items tend to become familiar to students over time (especially with increasing global access), maintaining novelty may require periodic updates to the item bank. This is particularly important for an inventory intended to be cross-national and enduring in its utility. Despite these challenges, a well-structured pool of adaptable contextual problems can balance authenticity with longevity. Such a redesign would still align the inventory with the principles of the APOS framework and retain its mathematical rigor while still enhancing its utility for engineering and other STEM educators.

As alternative assessment instruments, the Function Model [9] and the Precalculus Concept Assessment [8] provide valuable frameworks that that help inform the development of a skills-oriented inventory. The Function Model is built around four core competencies: modeling, interpreting, translation, and reification. Grounded in a "problem-solving environment" with contextually rich questions, it concisely captures the essential aspects of functions. Similarly, the Precalculus Concept Assessment focuses on three core reasoning abilities: the process view of functions, covariational reasoning, and computational reasoning, with questions designed to probe each of these areas. While the conciseness of these models and their development around core competencies are valuable, and their alignment with our aim to develop a skills-oriented inventory are clear, we believe they could be expanded to include additional competencies, such as reasoning with transformations and engaging with multiple representations. Additionally, skills such as covariational reasoning, focusing on how students visualize changes in a function's output relative to its input (as emphasized in the Precalculus Concept Assessment), could serve as specific extensions to the existing four competencies in the Function Model. In this context, covariational reasoning constitutes part of the interpreting competency, and explicitly assessing it would provide a more nuanced evaluation of students' ability to interpret in a way that is not presently addressed by the Function Model. Further, refining these with dynamic questions customized to multiple engineering contexts and use more modern survey techniques (e.g., adaptive surveys, coupled responses and qualitative elements) would enrich these models, making them more versatile for skill-focused assessment that emphasize flexibility, reasoning, and application in engineering contexts.

Core Mathematical Skills

Table 1 below highlights the core mathematical skills essential for developing an expert-level understanding of functions, as identified through preliminary scoping.

Table 1. Description of essential mathematical skills and their application to functions.

Skill	Description	Application to functions
Perspective Flexibility	Drawn from the idea of mathematical flexibility [15], this refers to the ability to shift specifically between perspectives or mental conceptions	• Manifests as the ability to transition selectively between viewing functions as actions (e.g., evaluating function values at specific inputs), processes, and complete objects on which further transformations can be applied • Progress along the APOS continuum indicates a student navigating thresholds, with reification recognized as a key troublesome aspect [5-7]
Visualization [7,8]	The ability to effectively construct, interpret and manipulate representations that accurately capture mathematical relationships	• Sketching of functions • Mentally mapping the effects of parameter changes on a function's graphs • Instrumental in reducing abstractions
Representational Translation [7-10]	The ability to seamlessly shift between different representational modes (e.g., tables, equations, and graphs)	• Characterises the troublesome nature of functions • Integrating information from multiple representations simplifies problem-solving by leveraging the most effective representational mode for a given task
Mathematical Language	The ability to accurately communicate mathematical ideas through symbols, concept-specific notation and terminology	• Symbolic literacy is a core component which demands understanding of function notation (e.g., those of inverse, and composite functions) • Definitional clarity particularly with function properties such as domain and injectivity
Mathematical Modeling [9]	The ability to formulate, use, interpret, and evaluate a mathematical model (the model being a mathematized abstraction of a real-world problem)	• Understanding that contextualizing is merely another, advanced mode of representation [8,10] • Handling of functions in contextualised problems by interpreting them within a mathematical framework and relating the solutions to real-world scenarios (e.g., function maxima interpreted as the highest point reached by a projectile) • Can be integrated easily into engineering contexts through problems of optimization
Abstraction	The ability to generalize and simplify complex functions by focusing on their essential properties and relationships rather than on specific instances	• For example, the ability to use the general form of a quadratic to derive its roots in terms of coefficients a, b, and c, rather than evaluating them for specific values • This principle of computational thinking allows for generalization and simplification

		• In modeling, translating real-world problems into mathematical terms exemplifies representational abstraction, where non-essential details are shed during the mathematization process
Covariational Reasoning	The ability to interpret and analyze the changing nature of one quantity in relation to another within dynamic situations [7,8]	• Enables students to perceive functions as dynamic relationships, instead of just static equations • Ensures progression from action to a process conception of functions, as underscored in [8] • Supports the embedding of contextual scenarios (e.g., how the height of a projectile varies over time) • Supports interpretation of rates of change, enriching understanding of function behavior such as growth, decay, and oscillation
Decomposition	The ability to break down a large, complex mathematical problem into smaller, more manageable sub-tasks	• A fundamental principle of computational thinking • Evident in problems requiring the deconstruction of a complex function into its base and the subsequent reconstruction through a series of transformations • A skill particularly transferable to engineering applications, where boundary conditions are utilized as markers to segment problems into distinct regions for simplified or advanced treatments
Concurrent Thinking	A pattern of thinking that involves the ability to make progress on multiple aspects at the same time	• In this context, this may relate to working with multiple representations of a function simultaneously • Unlike representational translation, this requires concurrent execution, such as forming a mental sketch of a function while computing its general shape, roots, and turning points

Concluding Remarks and Future Work

In summary, this Work-in-Progress paper emphasizes the need for a mathematical skills inventory tailored to undergraduate engineering students to address the limitations of traditional concept inventories by focusing on practical skills, competencies, and cognitive abilities. Using the function concept to motivate this notion, it underscores the importance of emphasizing core mathematical skills, such as perspective flexibility, visualization, and covariational reasoning, while emphasising their application in engineering contexts. Drawing insights from existing frameworks like the Function CI, Precalculus Concept Assessment, and Function Model, the proposed skills-based inventory aims to bridge the gap between theoretical understanding and practical application, equipping educators with insights to address skill deficits, enhance instruction, and support engineering students in navigating mathematical threshold concepts.

Future work will shift toward the development of a skills-based inventory targeting a single, pertinent mathematical skill (one that extends across multiple mathematical concepts such as functions, typically covered by first-year engineering students). This focused approach will allow for more refined assessment design and precise evaluation of students' understanding (and proficiency) within a specific cognitive domain. Building on the existing scoping review, the selected skill will be mapped against instructional needs and curricular goals within first-year undergraduate engineering mathematics courses. The development phase will engage faculty as integral stakeholders, ensuring the inventory provides actionable insights, is user-friendly and scalable for widespread implementation. Pilot studies will be conducted within engineering mathematics courses at two cross-national leading research-intensive institutions, Queen Mary University of London in the UK and Cornell University in the US. These studies will iteratively refine the instrument to improve its effectiveness and ensure its adaptability for use across a broader range of institutions.

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