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A visual and intuitive approach to teaching and learning the constant e and the function exp(x)

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WIP: A Visual and Intuitive Approach to Teaching and Learning the Constant e and the Function e^x

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Abstract

As technology becomes an integral part of daily life, students are adopting new ways of learning and increasingly favor visual, intuitive, and interactive methods. They are exposed to a wide array of videos, websites, and applications that enable them to explore topics, often without relying on traditional textbooks. These modern tools are perceived as more engaging and informative compared to conventional lectures and reading materials. To align with these evolving learning preferences, educators must adapt their teaching approaches.

In fact, in mathematics, traditional textbook-style lessons often present concepts in a highly abstract manner. One area that could greatly benefit from a more intuitive and visual approach is the mathematical constant e and the function e^x. Many students struggle to grasp an intuitive understanding of e^x, often missing its unique and elegant properties—such as the fact that its derivative is also e^x (as is its integral, up to a constant)—and its connection to fundamental processes like growth and decay.

The constant e and the function e^x frequently appear in both mathematical theory and real-world applications, playing a critical role in describing numerous STEM-related phenomena. Developing a deeper, more practical understanding of this function would empower students to apply their knowledge more effectively when solving problems beyond the confines of mathematics textbooks.

To address this, we propose a new approach to introducing e and e^x by emphasizing visualization to foster a more intuitive understanding of the function. In this paper, we explore key features of e^x, such as its derivative and integral, presenting them through visual representations that deepen students' comprehension. Highlighting these features can also help students apply the e^x function to novel problems they encounter.

Furthermore, we discuss real-world applications of e^x, demonstrating how it is used to model processes such as growth and decay. This approach offers a clearer

and more accessible understanding of the function, moving away from abstract explanations. Instructors may choose to incorporate these materials into their lessons to achieve similar outcomes.

This paper should be viewed as a work in progress. The material presented is not intended to replace any existing curriculum or textbook chapters but rather to serve as a supplementary resource, offering a deeper and more intuitive understanding of the concepts.

The content was introduced to students in three classes, followed by a detailed questionnaire: 25 students in the undergraduate course "Circuits 1," 43 students in the undergraduate course "Stochastic Models for CS," and 8 students in the graduate-level course "Modern Control." The feedback from students was overwhelmingly positive. They emphasized the value of visual, intuitive, and engaging methods of learning, which they found far more effective and appealing than traditional lectures or textbook-based approaches.

1 Introduction

With the increasing use of technology, students now rely on educational websites, YouTube videos, and other digital tools to learn concepts. These technology-based approaches are often more engaging and interactive compared to traditional methods, such as reading from textbooks alone. Students today expect to learn in ways that are more visual, intuitive, and connected to real-life applications. Mathematics, being one of the more abstract subjects, can particularly benefit from approaches that make its concepts more accessible and relatable. By presenting mathematical topics in a visual and intuitive manner, students can develop a deeper understanding of the material and build upon that foundation effectively.

Mathematical education thrives when visual and intuitive approaches are employed, particularly in conveying abstract and foundational concepts. Traditional, formula-driven methods often fail to instill a deep understanding, instead encouraging rote memorization. Notable works, such as those by Apostol and Mamikon [1], have demonstrated the value of using visual techniques to explain complex mathematical ideas without heavy reliance on symbolic manipulation. Tools like GeoGebra [2] and platforms such as Khan Academy [3] further illustrate the potential of interactive and visually engaging methods to make mathematical learning more accessible. Additionally, Grant Sanderson's work through 3Blue1Brown, leveraging the Manim animation library, exemplifies how dynamic visualizations can transform the understanding of mathematical relationships [4]. These resources underline the growing need for teaching strategies that align with students' preferences for intuitive, visual, and engaging methods, fostering a more meaningful connection with mathematical concepts.

Fortunately, there is already a wealth of online resources that present mathematical concepts in an engaging and approachable way. Examples include *Better-*

Explained [5] which provide valuable insights and innovative methods for reasoning about mathematical topics. These resources add tremendous value to the learning experience and serve as excellent examples of how abstract subjects can be taught effectively.

In this paper, we present a new approach to introducing e and e^x by emphasizing visualization, intuition, connections to nature, and real-life applications. This approach aims to foster a deeper understanding of the e^x function while also demonstrating its practical use in real-world scenarios. We highlight key features of e^x, such as its derivative and integral, using visual representations to enhance students' comprehension of e and its unique properties. Additionally, we explore real-world applications of e^x, such as modeling processes like charging a mobile phone, to show its relevance beyond theoretical mathematics. This approach strives to move away from abstract explanations and provide a clearer, more accessible understanding of the function.

This paper represents work in progress. The material presented is not intended to replace existing textbook chapters but rather to serve as a supplementary resource that instructors may choose to incorporate into their lessons. It is our hope that these materials will help educators achieve similar positive outcomes in their classrooms.

The content was introduced to students in three courses, followed by a detailed questionnaire: 25 students in the undergraduate course "Circuits 1," 43 students in the undergraduate course "Stochastic Models for CS," and 8 students in the graduate-level course "Modern Control." Feedback from students was overwhelmingly positive. They highlighted the value of visual, intuitive, and engaging learning methods, which they found far more effective and appealing than traditional lectures or textbook-based approaches.

2 Summary of Traditional Ways of Explaining e

2.1 Origin and History

The number e, approximately equal to 2.718, was not "discovered" in the sense of a single moment or person, but rather emerged over time through the work of several mathematicians. Jacob Bernoulli is often credited with the earliest recorded appearance of e (1683). He encountered the number while studying compound interest. This led to the realization that e arises naturally in processes involving growth and decay. Leonhard Euler formally defined the number e and explored its properties, including proving that e was an irrational number (1730s). He showed that e is the base of the natural logarithm and extensively studied its applications in calculus.

2.2 What is e ?

e is an irrational number that can be approximated by 2.71828 and is the base of the natural logarithm and exponential function. It can be defined as the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

an expression that arises in the computation of compound interest.
It is the sum of the infinite series

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots.$$

There are many other representations of e, for example those shown in [6] [7].

3 On the derivative of e^x

A great explanation about the uniqueness of e^x. Can be found in the you tube video from *blackpenredpen* [8]. The video clearly explains why the derivative of e^x is equal to itself, i.e., to e^x. Simply put, The slope of a line tangent at any point x is also equal to e^x. It shows that this is not the case with say 2^x or 3^x.

4 Our Approach

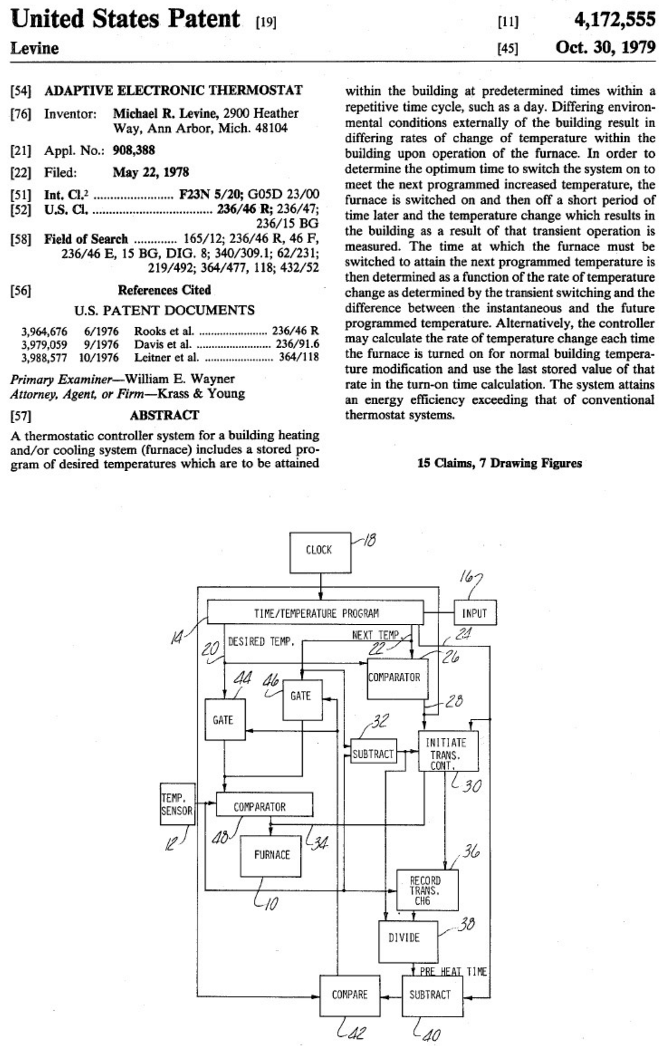
4.1 A Short Story

In order to excite students, we started with a short story that the students could relate to, and see how applications could be built using a function of e over time. In the late 1970s and 1980s, Michael R. Levine created multiple inventions related to digital thermostats, utilizing components such as capacitors and resistors alongside equations involving e, such as Newton's Law of Cooling [9]. This law states that the rate of temperature change of an object ($\frac{dT}{dt}$) is proportional to the temperature difference between the object (T) and its surroundings (T_{env}):

$$\frac{dT}{dt}=-k(T-T_{\text{env}})$$
$$T(t)=T_{\text{env}}+(T_0-T_{\text{env}})e^{-kt}$$

where T_0 is the initial temperature of the object, and k is a positive constant. The solution shows that temperature follows an exponential decay, gradually approaching the ambient temperature over time.

He created multiple patents that described how they could be used to replace the analog thermostats that were currently in use. One of his patents *Adaptive electronic thermostat US 4172555* [10] was bought by Honeywell, and the ideas are being used these days in most digital thermostats. He literally affected many peoples' lives. The following is a snapshot of the patent's first page (Figure 1).



United States Patent [19] — Levine

[11] 4,172,555

[45] Oct. 30, 1979

[54] ADAPTIVE ELECTRONIC THERMOSTAT

[76] Inventor: Michael R. Levine, 2900 Heather Way, Ann Arbor, Mich. 48104

[21] Appl. No.: 908,388

[22] Filed: May 22, 1978

[51] Int. Cl.² F23N 5/20; G05D 23/00

[52] U.S. Cl. .... 236/46 R; 236/47; 236/15 BG

[58] Field of Search .... 165/12; 236/46 R, 46 F, 236/46 E, 15 BG, DIG. 8; 340/309.1; 62/231; 219/492; 364/477, 118; 432/52

[56] References Cited

U.S. PATENT DOCUMENTS

3,964,676	6/1976	Rooks et al.	236/46 R
3,979,059	9/1976	Davis et al.	236/91.6
3,988,577	10/1976	Leitner et al.	364/118

Primary Examiner—William E. Wayner
Attorney, Agent, or Firm—Krass & Young

[57] ABSTRACT

A thermostatic controller system for a building heating and/or cooling system (furnace) includes a stored program of desired temperatures which are to be attained within the building at predetermined times within a repetitive time cycle, such as a day. Differing environmental conditions externally of the building result in differing rates of change of temperature within the building upon operation of the furnace. In order to determine the optimum time to switch the system on to meet the next programmed increased temperature, the furnace is switched on and then off a short period of time later and the temperature change which results in the building as a result of that transient operation is measured. The time at which the furnace must be switched to attain the next programmed temperature is then determined as a function of the rate of temperature change as determined by the transient switching and the difference between the instantaneous and the future programmed temperature. Alternatively, the controller may calculate the rate of temperature change each time the furnace is turned on for normal building temperature modification and use the last stored value of that rate in the turn-on time calculation. The system attains an energy efficiency exceeding that of conventional thermostat systems.

15 Claims, 7 Drawing Figures

Figure 1: US Patent: Adaptive Digital Thermostat

4.2 Visualizing e^x

A. Exploring the Beauty of e^x

The function e^x is a mathematical expression, and visualizing it highlights its elegance while aiding comprehension.

B. Slope and Height

Let's begin with a fundamental visualization: The graph of e^x reveals a remarkable property—at any point x, both the value of the function and the slope of its tangent line are equal to e^x.

C. A Practical Analogy

Imagine a person climbing a hill whose slope increases exponentially, following the function e^x (Figure 2). If the climber knows only the slope at any point, they can deduce their current elevation—it is equal to the slope! As the elevation rises, the slope increases at the same rate.

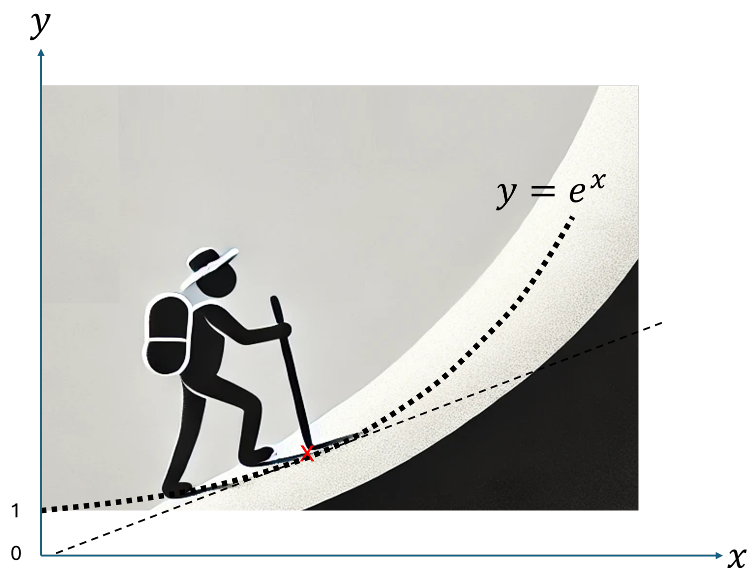


Figure 2: A Person Climbing a Hill

D. Height, Slope, and the Area Under the Curve

One of the most fascinating aspects of e^x is that its value at any point is simultaneously:

1. The height of the curve.
2. The slope of the tangent line.
3. The area under the curve (its antiderivative).

This counterintuitive property makes e^x unique and visually striking when presented on a graph (Figure 3).

Now consider two distinct points, x_1 and x_2, on the same e^x curve (Figure 4). Building on the visualization from Figure 3:

- The difference in the function values at these points, $e^{x_2} - e^{x_1}$, equals the difference in the slopes at x_2 and x_1.
- Additionally, the area under the curve between x_1 and x_2 (highlighted in blue in Figure 4) is also equal to $e^{x_2} - e^{x_1}$.

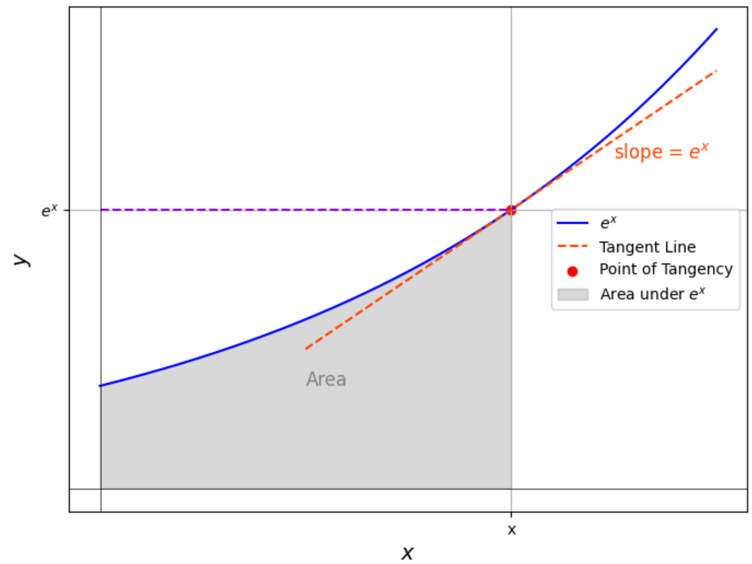


Figure 3: Height, Slope and Area

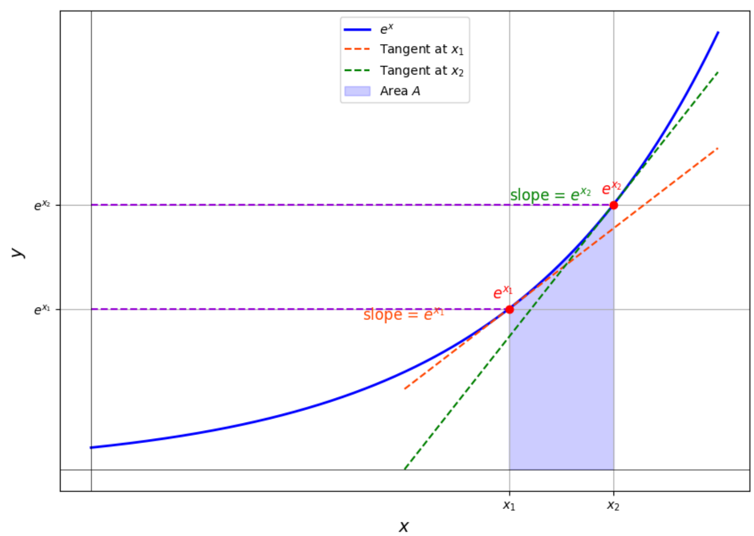


Figure 4: Difference in Function's Values

This remarkable property connects the function's value, slope, and integral in a visually and intuitive ways.

The same concept can be extended to functions with different exponents. While the specific results may differ, the core idea remains unchanged: there is a clear and intrinsic relationship between the function's value, its slope, and the area under the curve.

This presents an excellent exercise for students: explore and determine the relationships between the function, its slope, and the area under the curve when the exponent is nx instead of x (Figure 5). Analyzing this case helps deepen the understanding of these interconnected properties in exponential functions.

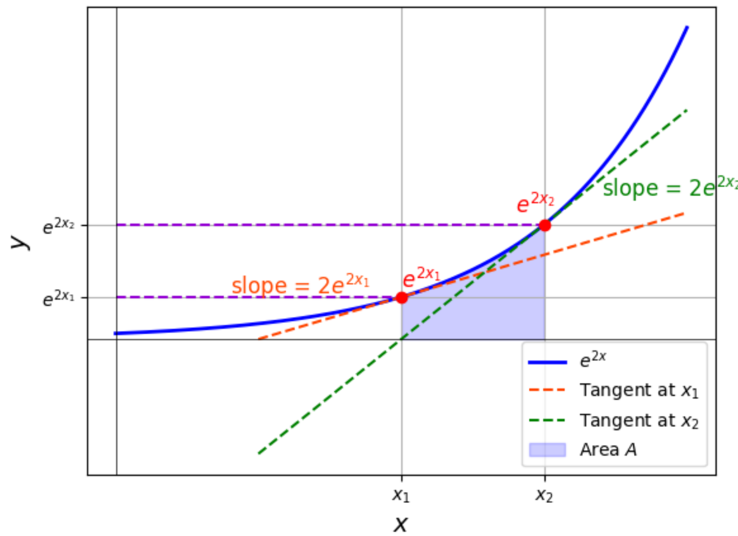


Figure 5: e^{2x} Function

4.3 Familiar Applications

A. Cell Phone Charging

Another important application of e^x involves visualizing exponential decay, which occurs when x is negative. A highly relatable example is modeling phone battery charging, where x represents time (t).

This example is particularly effective due to students' familiarity with mobile devices [11]. Refer to Figure 6.

The rate at which a battery charges depends on its current charge level—the higher the charge level, the slower the rate. At $t = 0$ the charging rate is at its peak; at $t = \tau$ the rate decreases; and at $t = 2\tau$ the rate is even lower. This pattern can be observed in the slope of the curve at different time intervals. Assuming the phone's internal circuit is an RC circuit, the charging process follows a function characterized by exponential decay.

B. Diffusion

Diffusion occurs exponentially as molecules move from high to low concentration. This natural process happens as molecules randomly collide, spreading into open spaces rather than staying crowded. For instance, in two compartments separated by a wall—one with gas and the other a vacuum—the rate depends on factors like

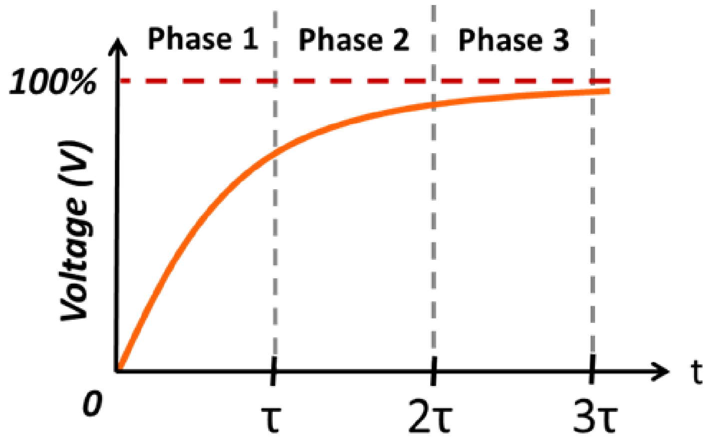


Figure 6: Finding Slope of Charging Rate at 3 points

pressure difference, initial concentration, and opening size. Over time, concentrations in both chambers equalize exponentially. See Figures 7, 8 and 9.

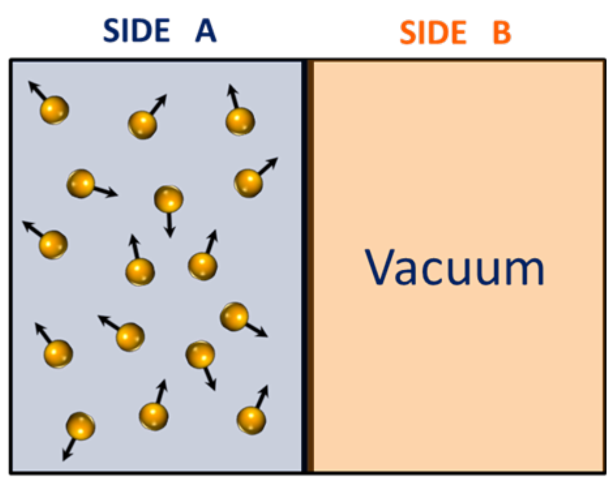


Figure 7: Diffusion "Before"

4.4 Engaging Puzzle

We used a brainteaser on exponential decay to engage students [11].

To help students connect with the concept of exponential decay, we used relatable examples and interactive elements. One example involved the cooling of coffee - a common experience where hotter coffee cools faster initially and gradually approaches room temperature. (See Figure 10).

We presented a puzzle to solidify understanding. Students were asked: *If you add cold cream to one cup of coffee immediately and to another cup after ten minutes, which cup will be cooler after ten minutes*?

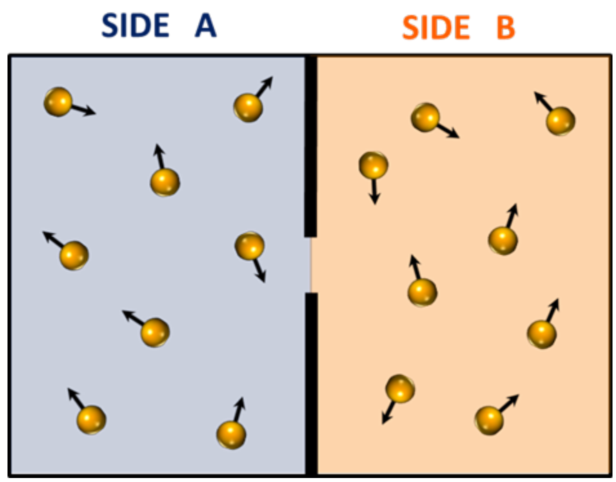


Figure 8: Diffusion "After"

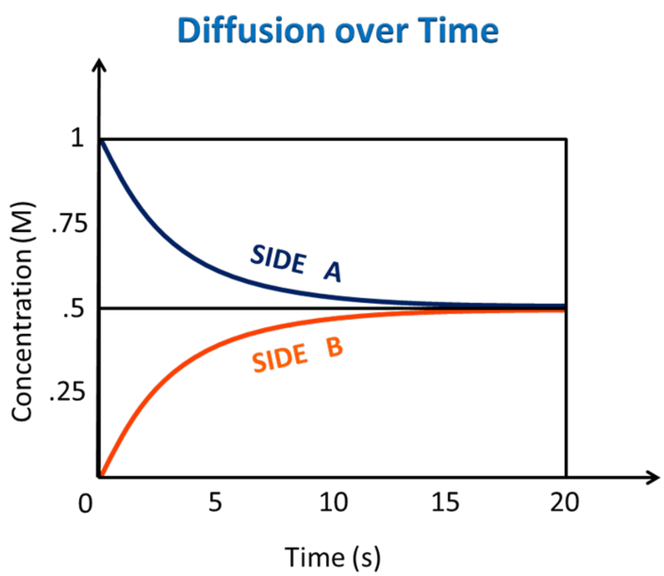


Figure 9: Diffusion Over Time

The answer : the cup where cream was added later will be colder. This is due to exponential decay—since the second cup starts at a higher temperature, it cools faster initially.(See Figure 11)

5 Assessment

To evaluate student perceptions of the e^x approach and gather feedback on teaching methods, we implemented the approach in three different courses. This was followed by a detailed, anonymous questionnaire. The courses included are:

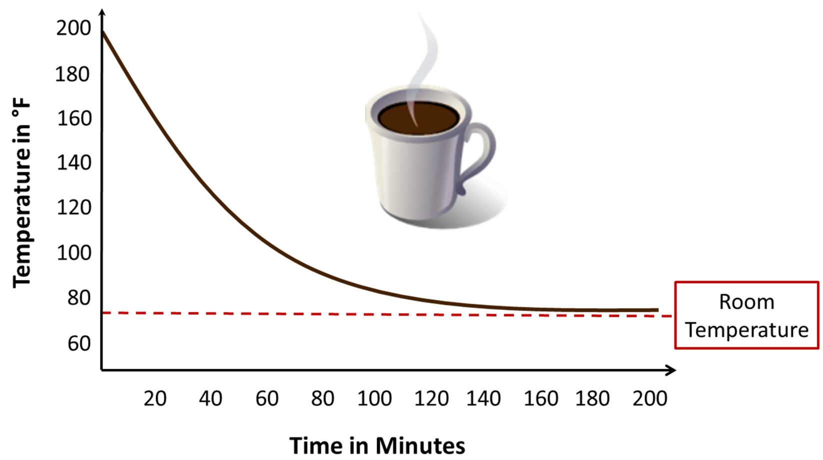


Figure 10: Coffee Temperature Decay

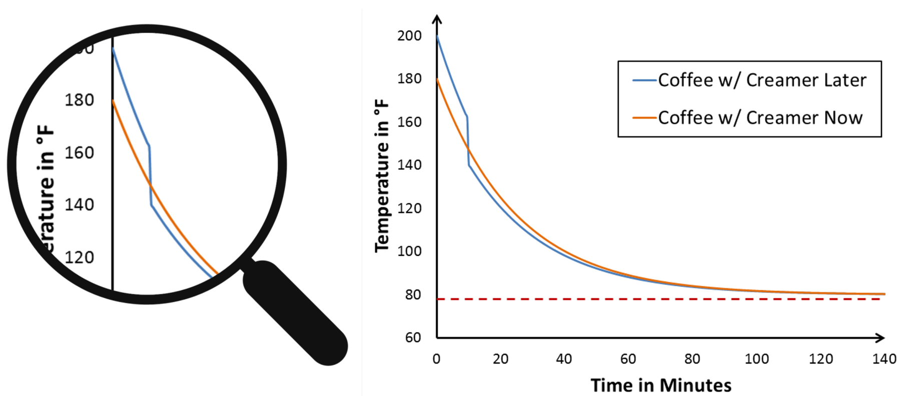


Figure 11: Temperature as a function of time.

- "Circuits 1" (25 undergraduate students)
- "Stochastic Models for CS" (43 undergraduate students)
- "Modern Control" (8 graduate students)

The questionnaire and detailed results are provided in the Appendix.

Feedback from students was overwhelmingly positive. They emphasized the value of visual, intuitive, and engaging teaching methods, which they found significantly more effective and appealing than traditional approaches such as lectures or textbook-based learning.

Key Findings

1. Importance of Visual Learning

- Questions related to visualizing concepts consistently scored above "important" on average across all courses.
- Students highlighted visual methods as critical to their understanding, particularly in contrast to traditional tools like PowerPoint slides, textbook readings, and lectures, which scored lower in perceived effectiveness.

2. Student Engagement and Preferences

   - Most students rated their mathematical understanding as above average.
   - High engagement levels were reported during class exercises involving e^x, reinforcing the importance of active learning methods.
   - Students showed a clear preference for guided, in-class learning over independent study or textbook-based approaches.

3. Graduate vs. Undergraduate Preferences

   - While undergraduate students showed a notable decrease in preference for self-teaching, graduate students rated this slightly higher. This distinction suggests differences in learning preferences based on academic level.

In short, one of the most significant takeaways was the strong emphasis on visual learning for e^x. Across all courses, visualization questions consistently received high scores, demonstrating the effectiveness of this approach in enhancing comprehension and engagement. While the question of self-directed learning was beyond the primary scope of this study, the observed differences between undergraduate and graduate students warrant further investigation.

6 Conclusion

In this paper, we explored the unique aspects of the constant e and the function e^x, presenting them to students through visual, intuitive, and engaging methods.

To make these abstract concepts more accessible, we showed the remarkable property of e and e^x in a single graph: the function, its derivative, and its integral all share the same value at every point. This property deepens students' understanding of the mathematical elegance and consistency of the function.

Additionally, we introduced relatable applications such as phone charging and coffee cooling. These examples allowed students to better understand how the function connects to real-world phenomena. We also discussed a particularly well-received application: a successful thermostat-related patent based on the exponential decay of temperature over time. This patent demonstrates how e^x can be applied to practical problems, highlighting its potential for solving real-world challenges.

It's important to note that this approach is not intended to replace traditional lessons but rather to serve as an introductory addition to any course. The aim is to promote a deeper, more visual understanding of e and e^x. We are encouraged by the positive feedback from students, who appreciated the engaging and intuitive nature of the presentation.

Acknowledgments

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References

[1] Tom M. Apostol. “A Visual Approach to Calculus Problems”. In: *Engineering and Science* 63.3 (2000), pp. 22–31. URL: https://resolver.caltech.edu/CaltechES:63.3.Calculus.

[2] GeoGebra. *GeoGebra: Interactive Mathematics Software*. 2023. URL: https://www.geogebra.org/.

[3] Khan Academy. *Khan Academy: Free Online Courses, Lessons and Practice*. 2023. URL: https://www.khanacademy.org/.

[4] Grant Sanderson. *3Blue1Brown YouTube Channel*. 2024. URL: https://www.youtube.com/c/3blue1brown.

[5] Kalid Azad. *An Intuitive Guide To Exponential Functions and e*. Accessed: 2025-01-10. 2007. URL: https://betterexplained.com/articles/an-intuitive-guide-to-exponential-functions-e/.

[6] Jon Rogawski and Colin Adams. *Calculus*. 3rd. New York: W.H. Freeman, 2015. ISBN: 978-1-4641-2526-3.

[7] Laurence D. Hoffmann and Gerald L. Bradley. *Calculus for Business, Economics, and the Social and Life Sciences*. 10th. New York: McGraw-Hill, 2009. ISBN: 978-0-07-353231-8.

[8] blackpenredpen. *Why is the derivative of e^x equal to e^x?* https://www.youtube.com/watch?v=oBlHiX6vrQY. Accessed: 2025-01-16.

[9] J. H. Lienhard V and J. H. Lienhard IV. *A Heat Transfer Textbook*. 6th. Version 6.00. Cambridge, MA: Phlogiston Press, Apr. 2024. 810 pp. URL: https://ahtt.mit.edu.

[10] Michael R. Levine. “Adaptive electronic thermostat”. US4172555A. Filed May 22, 1978; Issued October 30, 1979. Oct. 1979. URL: https://patents.google.com/patent/US4172555A/en.

[11] Daniel Raviv and George Roskovich. *Understood: A Visual and Intuitive Approach to Teaching and Learning Control Systems*. n.p.: CreateSpace Independent Publishing Platform, 2014. ISBN: 9781500288884.

Appendix

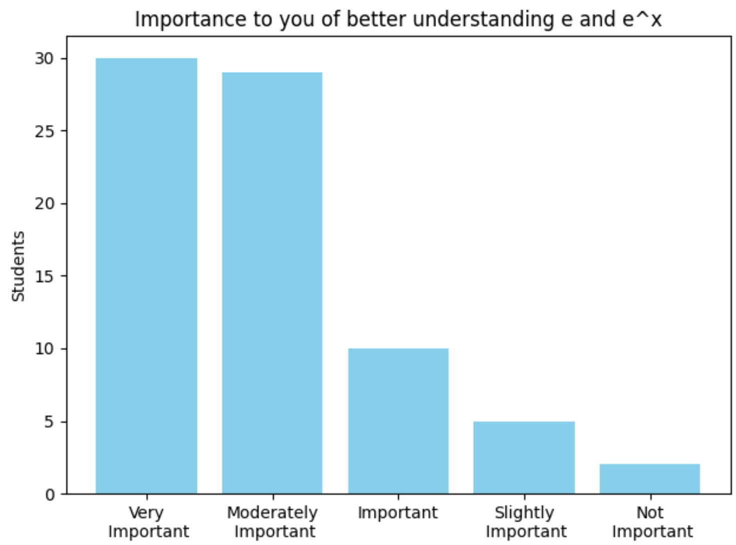

Figure 12: Student Feedback about the Importance of understanding e and e^x

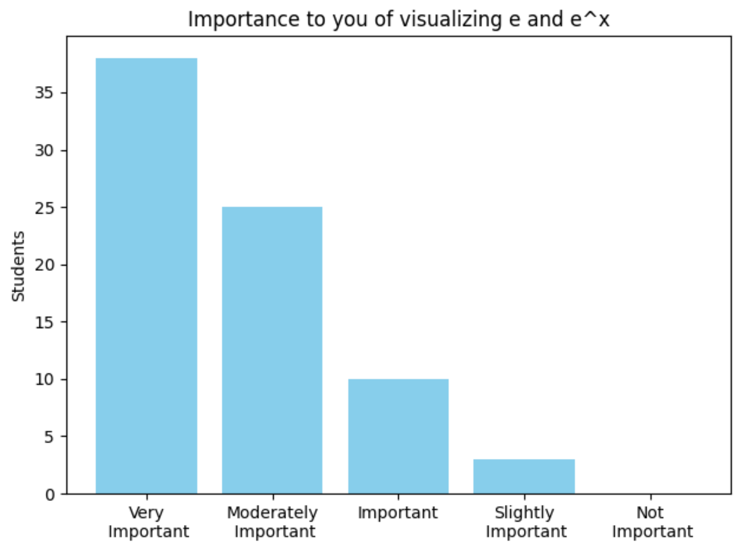

Figure 13: Student Feedback about the Importance of Visualizing e and e^x

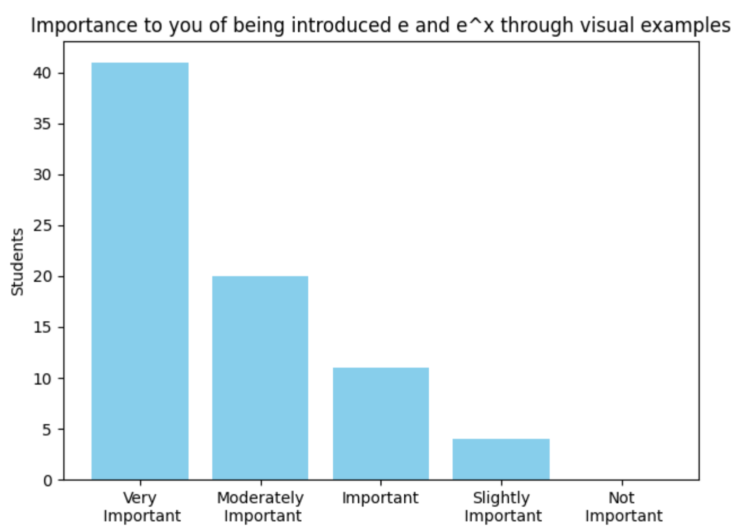

Figure 14: Student Feedback about the Importance of visual examples

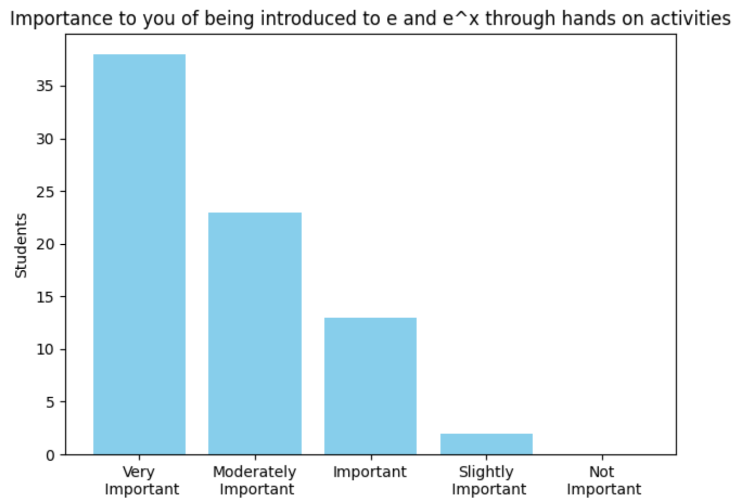

Figure 15: Student Feedback about the Importance of Hands-on Activities

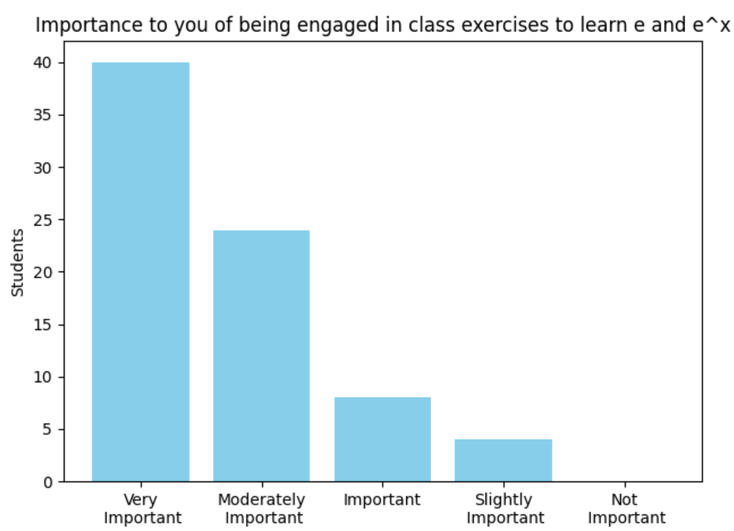

Figure 16: Student Feedback about the Importance of Class Exercises

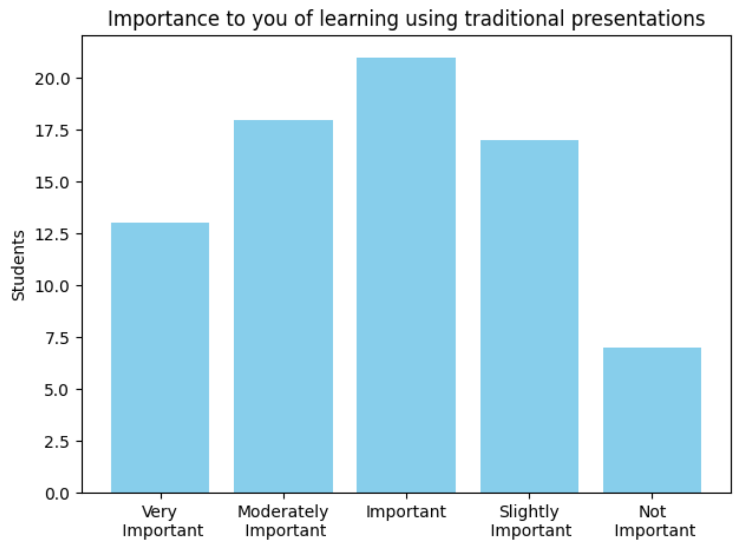

Figure 17: Student Feedback about the Importance of Traditional Presentations

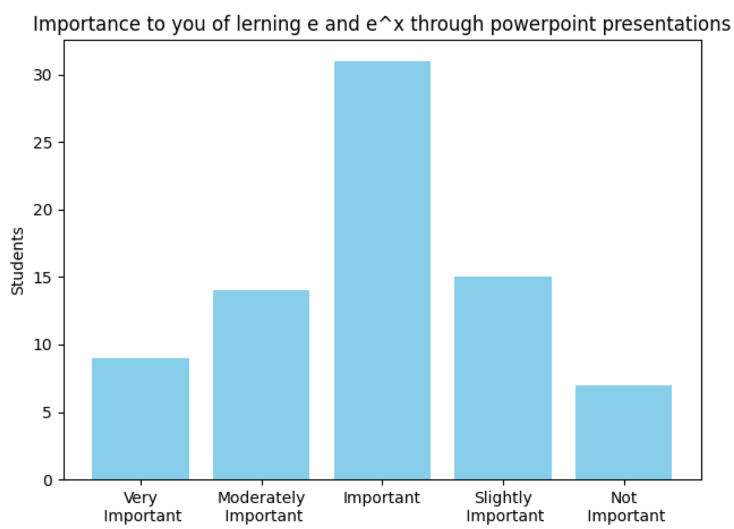

Figure 18: Student Feedback about the Importance of PowerPoint

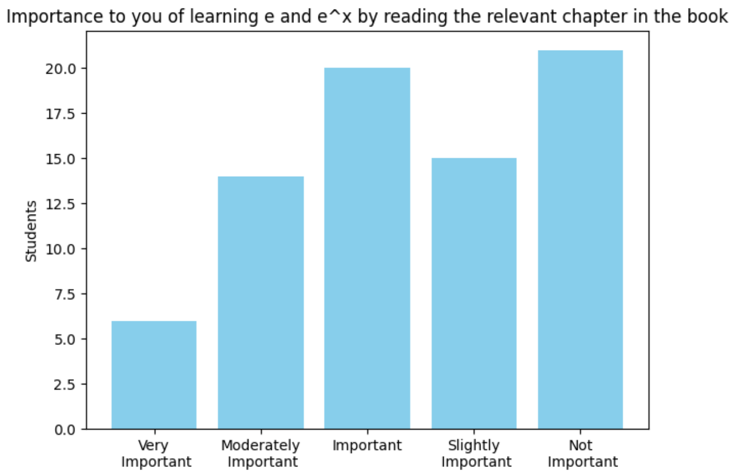

Figure 19: Student Feedback about the Importance of Readings

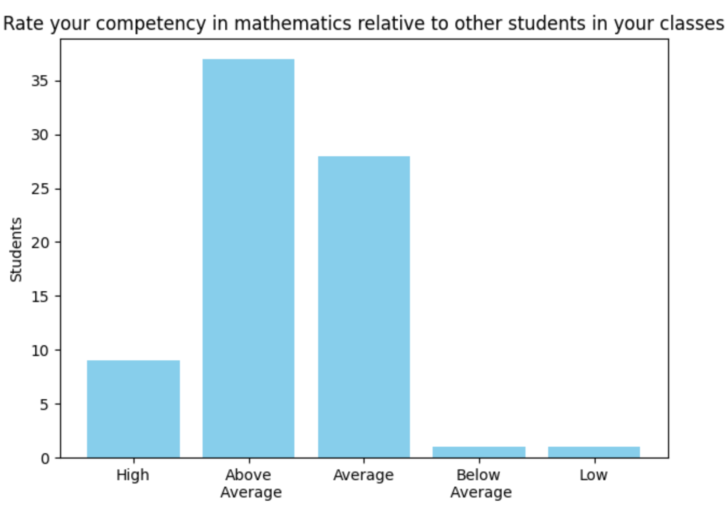

Figure 20: Self-Reported Competency Distribution

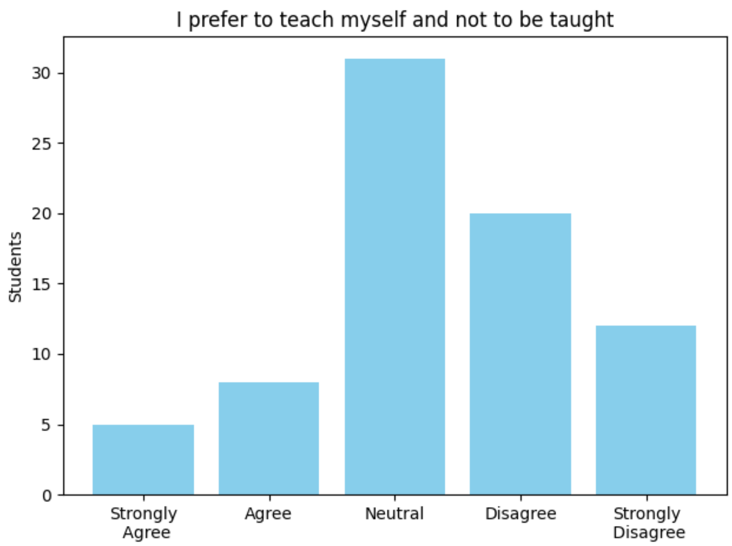

Figure 21: Student Feedback on if they Prefer to Self Teach