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Equivalent Systems, Distributed Loads, and Reactions - One Model to Teach Them All

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Equivalent Systems, Distributed Loads, and Reactions
One Model to Teach Them All

Abstract

Equivalent systems, distributed loads, and reactions are often treated as highly theoretical and largely disconnected concepts by Statics students. Integrated learning objectives, basic theory, real world scenarios, and low-cost 3D printed beam model demonstrations connect all three major topics. The beam model employs two low-cost kitchen scales, a small set of gram weights, and a 3D-printed, simply supported beam model. In the first class, after the instructor introduces equivalent systems, students set up a particular loading scenario with multiple weights at several locations along a beam. They then calculate the resultant force and moment and the resultant force at a location, noting that for each of the three equivalent systems the kitchen scales show the same reactions. In a second class on distributed loads, students approximate a triangular distributed load on the same model beam and then experimentally determine the location of the resultant force that results in an externally equivalent system (i.e. identical reactions). Students then compare this location to the location of the centroid of the distributed load. In a final class on reactions, students again model a unique multi-force load combination including both point and distributed loads. Students then compare the reactions predicted by the rigid body equations of equilibrium with the reactions measured on the scales. Using low-cost, hands-on models, students gain the opportunity to build better mental models associated with engineering analysis tools, to recognize the variations between idealized models and the natural variability of reality, and to engage with the challenges of comparing theoretical predictions with measured values. By reusing the same beam model multiple times, students draw connections between equivalent systems, distributed loads, centroids, and reactions. Real world scenarios encourage curiosity about the world and demonstrate how statics is an important first step toward creating value for others. Faculty eager to teach these topics will find comprehensive coverage of the topics and the use of the beam model to teach the topics. Thoroughly demonstrated applications of John Milton Gregory's *Seven Laws of Teaching* [1] should also be helpful to the engineering educator.

Notation

$\vec{F} =$	force
$F_{Ay} =$	y Cartesian component of a force reaction at point *A*
$F_{Bx} =$	x Cartesian component of a force reaction at point *B*
$F_{By} =$	y Cartesian component of a force reaction at point *B*
$F_x =$	x Cartesian component of a force, or force along the x axis
$F_y =$	y Cartesian component of a force, or force along the y axis
$\overrightarrow{F_R} =$	resultant force
$F_{Rx} =$	horizontal component of the result force
$F_{Ry} =$	vertical component of the result force
$\vec{M} =$	applied moment, moment couple, or force couple
$\overrightarrow{M_{/A}} =$	moment about point *A*
$\overrightarrow{M_{/B}} =$	moment about point *B*
$\overrightarrow{M_{R/A}} =$	resultant moment about point *A*
$\overrightarrow{r_{/A}} =$	position vector (or relative position with respect to point *A*
$x =$	horizontal coordinate for the location of a force
$y =$	vertical coordinate for the location of a force
$\rightarrow_x =$	scalar notation indicator in the x direction
$\uparrow^y =$	scalar notation indicator in the y direction
$\uparrow\uparrow^z =$	scalar notation indicator about the z axis

Introduction

An engineering statics course lays critical foundations for student engineers both in future engineering course sequences and their careers. Students must develop mastery over many topics including equivalent systems, distributed loads, and reaction calculations to succeed in dynamics, solids and fluids mechanics courses, not to mention structural and mechanical design courses. Yet these particular three topics are often taught as highly theoretical and disconnected from each other: equivalent systems might be explored to develop vector mathematics competency in "moving" forces and moments; distributed loads are often caught in a no-man's-land between equivalent systems and centroids; while rigid body equilibrium might be received as completely unique and connected to previous topics only by virtue of leveraging vector manipulation skills [2]. Yet, all three of these skills connect to one another. A reoccurring simply supported beam model can introduce, explore, and connect all three topics, while engaging students with real-world applications that create value and drive curiosity [3].

The low-cost, 3D printed beam model (hereafter, the beam model) can introduce students to equivalent systems, distributed loads, and rigid body equilibrium reactions. The beam model and activities are direct extensions of the work of Tonya Nilsson and others at Santa Clara University [4], and a great debit of thanks is owed them. The literature review explores benefits, educational theory, implementation approaches, and previous work developing physical models to teach statics concepts. The application section describes the implemented pedagogy and the beam model including its 3D-printed and low-cost, off-the-shelf components. The remainder of the paper describes the application of the beam model to teaching equivalent systems, distributed loads, and simply supported reactions including main ideas, learning objectives, value and curiosity creating contexts, model implementations, and example problems. The beam model is an alternative presentation method for the topics discussed; therefore, typical assessments remain valid and are outside the scope of this work. A further-work section identifying additional topic areas for model implementation follows a brief discussion on student reception and effectiveness.

Literature Review

Students in engineering mechanics courses develop many skills, chief being conceptual awareness of how the created world works. As they pursue the true, the beautiful and the good in an objective world, student engineers demonstrate their conceptual competence through concurring representational competence reflected in their descriptions, diagrams, and mathematics [5], [6]. However, far too often students revert to memorized responses to idealized contexts rather than rationally exploring real world conditions [7]. To this end, physical models in the classroom are a well-established method for developing conceptual competence, visualization skills, and critical thinking around complex realities [7], [8]. Physical models may support qualitative exploration, the analysis of basic applications and introductory problems, and/or the exercise of critical thinking on more complicated real world problems [5]. In statics topics alone, ASEE contributors have developed numerous demonstrations, experiments, and activities to explore connections [7], [8], trusses [9], area centroid and moment of inertia [10], [11], [12], pulleys and frames [13], distributed loads, resultants and superposition [4]. Other approaches include studio-style model creation by students [14] or comprehensive statics modeling kits [5], [15]. The beam model described here joins the ranks of these hand-on mechanics tools [16].

Many designers of physical models attempt to leverage a constructivist theory of education by helping students “create meaning from his or her own experiences” [17]. Constructivist educators have rediscovered active, student-centered, and conversational learning activities; however, engineers are not free to “create meaning”. They must align their understanding with an objective world [18]. Rather than “discover” and “interpret” to create their own reality with the Rousseau-inspired guide-on-the-side, engineering students should pursue the objectively true, beautiful, and good and embrace the generously-shared expertise of older scientist and engineers [19]. The sage-on-the-stage plays the role of the expert tour guide, provides generational insight into concepts long understood, and highlights the clear paths to the edge of human knowledge. While standing on the shoulders of giants, students contribute to the pursuit of truth and the design of the good and beautiful.

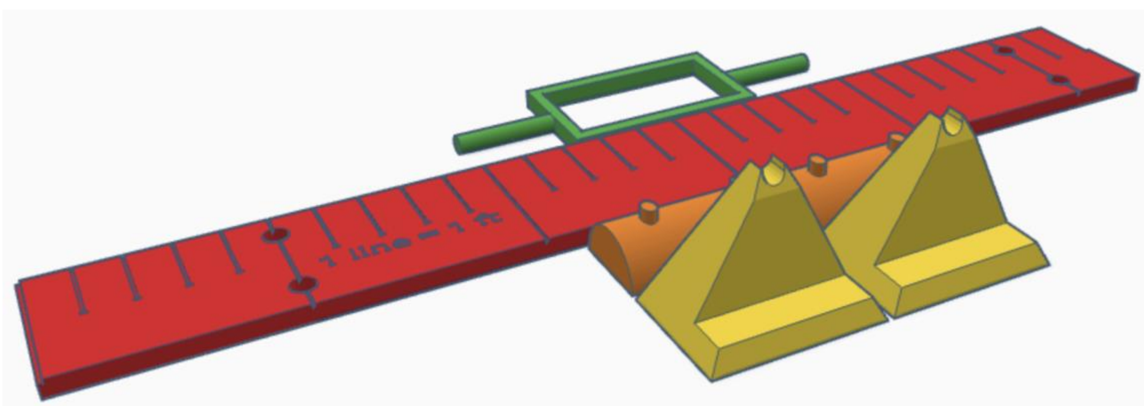
The literature describes many ways to incorporate physical models into the classroom ranging from pass-around aids in unidirectional lectures [20] to handout-driven flipped classrooms [4]. The order of operations for incorporating models also vary in the literature. Some educators begin by introducing calculations, verifying and exploring with the physical model, and then reflecting on lessons learned [4]. Other practitioners introduce the conceptual ideas with physical modeling exercise and then reinforce the concepts with calculations [14]. This paper advocates for instructor-guided experimentation and discussion.

Application

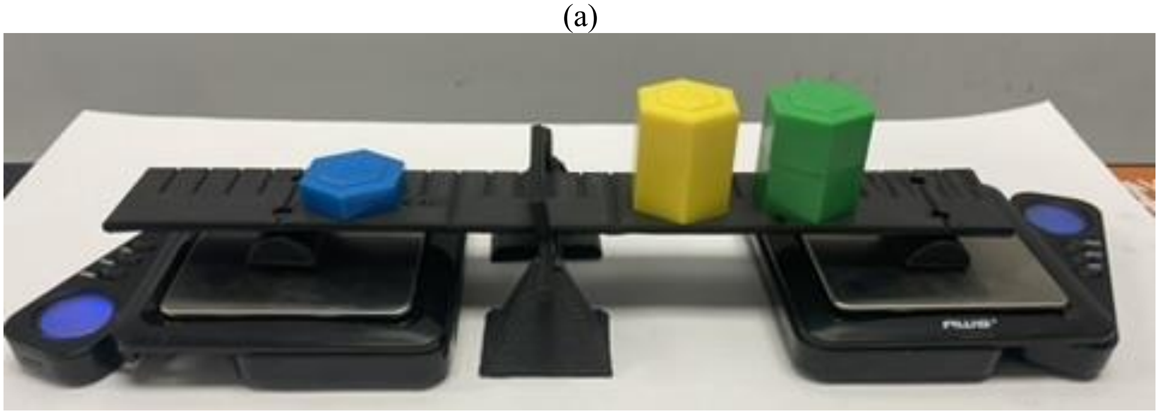
The path of learning is well-established within the classical western paideia where active learning and conversation reign [21]. The Classical Model of education suggest that students must first engage with the Grammar of a topic (definitions, components, etc.) before attempting Dialectical analysis and Rhetorical application[22]. For the purpose of these demonstrations, John Milton Gregory’s *Seven Laws of Teaching* is a sufficient guide [1], [23]. A recommended pattern begins with a ready instructor (Law 1) engaging the students’ attention (Law 2) with a value creating context [3] and introducing necessary vocabulary verbally, visually, and mathematically (Law 3). Using the physical model to connect the new concepts to what students already know (Law 4), the instructor helps the students build their own mental model of the desired concepts (Law 5). The students then practice thinking and expressing the specific modeled scenario in the terms (descriptive, pictorial, and mathematical) introduced at the beginning of the lesson (Law 6). After the demonstration, students reflect, review, rethink, reproduce and apply the concepts learned via traditional homework and/or other formative assessment (Law 7) [24]. The beam model demonstrations help students walk the problem solving path of interpreting reality using words, developing a visually-expressed mental model, and mathematically expressing an appropriately simplified idealization [6]. By solving the problem and checking for coherence between the analytical model, mental model and reality, students gain confidence in their analysis. By using the beam model to teach multiple topics during the semester, student cognitive load is decreased; once the basic structure of the beam model is understood, students can focus on understanding the engineering concepts, using the principle of “Agassiz and the Fish” [25].

The beam model itself consists of three major sets of components: gram weights, kitchen scales, and 3D-printed components. Low-cost K-8 level gram weights [26] generate loads for the demonstrations. These weights are hexagonal and stackable allowing for precise placement and variations in load. The provided example problems scale 1 g to 100 lb. Special care must be taken to delineate between the mass in grams and the "full-scale" equivalent weight (force) in lbs. Two small kitchen scales [27] measure the reactions for the beams also in grams. The selected scales are sized to interact well with the gram weights and the 3D printed components.

Figure 1a shows six pieces of the 3D-printed components [28]. The beam itself (in red) is 27 cm long with 1 cm increments intended to model 1 ft in the full-scale problems. Holes 20 cm apart allow for accurate placement on the roller supports (in orange). The beam and supports work together to form a simply supported beam. The green frame and yellow brackets are designed to induce pure moment to the beam. Sliding the green frame around the red beam between the orange supports and resting it on the yellow brackets allows the user to apply twist to the green frame resulting in equal and opposite forces on the red beam, separated by a perpendicular distance. The yellow brackets ensure no net force is applied to the beam. The brackets are sized to ensure proper alignment of the frame with the center line of the red beam, sitting on the orange supports on top of the kitchen scales as seen in Figure 1b.



(a)



(b)

Figure 1. The beam model including (a) rendering of 3D printed components and (b) the assembled model with kitchen scales, gram weights, and 3D printed components.

Equivalent Systems

Main Idea

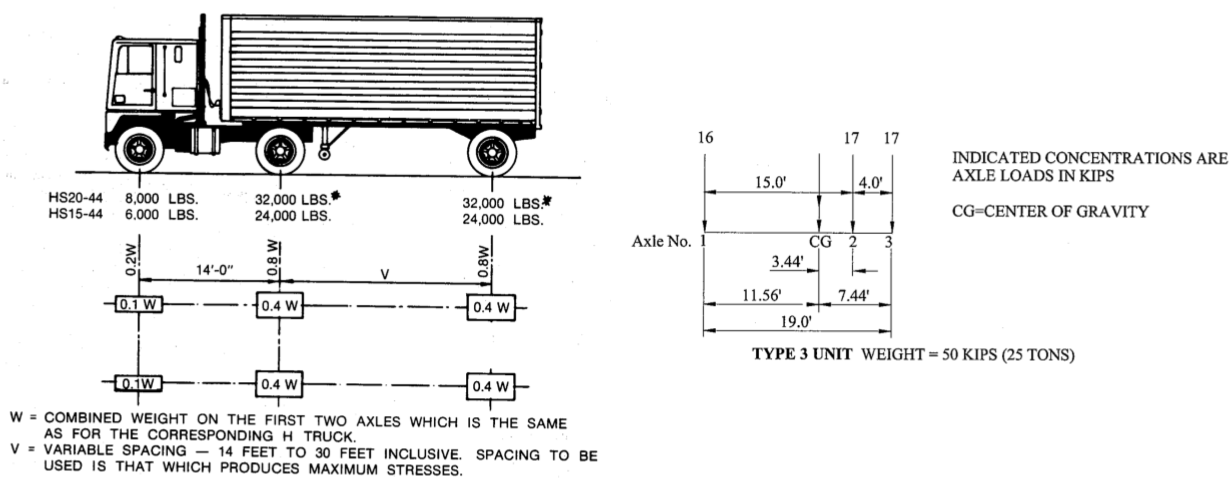
As statics students gain comfort with vector mathematics and vector representations of force and moments, they practice these skills by describing equivalent systems. By the conclusion of this exercise, student should be able to:

- Define equivalent systems.
- Classify three forms of equivalent systems.
- For force and moment loads, calculate an equivalent system of…
    - The resultant force and moment at a point.
    - The resultant force at a point.

Demonstration

Context: A New State Legal Load for Park Bridges

In the United States, highway bridges are designed according the *AASHTO LRFD Bridge Design Specification* [29]. This document specifies the standard design vehicle for highways as the HS-20 truck (see Figure 2a). The HS20-44 truck is intended to produce worst case loadings on bridges equivalent to the modern 80,000lb semi-tractor trailer (a.k.a. 18-wheeler). However, the design standard also allows for the states to adopt their own legal limit vehicles particularly for load posting of smaller bridges. For example, Figure 2b shows a Type 3 Unit equivalent to a typical fully loaded dump truck.



(a) (b)

Figure 2. (a) HS20-44 design truck and (b) Type 3 Unit Legal Load [29]

For this example, student engineers will consider equivalent loadings for a new proposed state legal load for use on park bridges. If this load is approved, it will allow construction companies working on park improvement projects to drive conforming vehicles on park pedestrian bridges such as those found in Greenville, SC (Figure 3).

(a)

(b)

Figure 3. Park bridges in Falls Park on the Reedy, Greenville, SC
(a) the Liberty Bridge [30] and (b) Eugenia Duke Bridge [31]

Student engineers will be evaluating a test bridge with the new proposed state legal load for park bridges shown in Figure 4. The load consists of a front axle caring 500 lb or 0.5 kips, a second axle 10 ft later carrying 2000 lb or 2 kips, and a third axle 4 ft later caring another 2000 lb. Student engineers must find two forms of simplified equivalent systems for this loading: 1) a force and moment applied at the lefthand roller A, and 2) a force applied a distance to the right of the left-hand roller A.

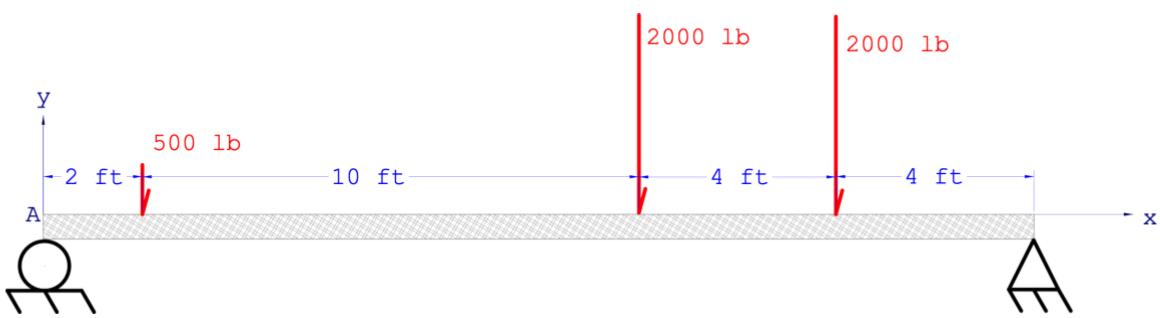


Figure 4. New proposed state legal load for park bridges places on a 20 ft simply supported test bridge. All distances measured in USCS feet, and all loads applied in 100 lb increments.

Modeling: Complex Loading

Student engineers, currently unfamiliar with the analytical tools used by practicing engineers, can start by creating a scale model of actual loading. The scaling factors use 1 g to model 100 lb and one space on the 3D printed beam (1 cm) to model 1 ft on the test bridge. Students are instructed to set up the beam model as shown in Figure 5. Students start by placing the scale, supports, and beam and then zeroing the scales. The students then place the weights for each axle load: the 5 g blue weight for the first axle load, the 20 g yellow for the second axle, and two 10 g yellow weights for the third axle load.

(a) (b)

Figure 5. Actual loading set up for equivalent system example (a) profile view and (b) plan view.

The instructor and students can then discuss their confidence that what they have set up is an accurate and/or precise model of the actual systems. What variations might exist in each set up, or in the accuracy of the measuring devices, etc.? If there are several models in the room (ideally a model for groups of two to four students), the instructor can then collect student data on the measured reactions. Measurements will not be identical, rather natural variability will occur around averages. Students can then discuss the variation: are the numbers wrong? If so, which one is right? Should the precision (typically 4-5 significant figures) on the scales be trusted? Are they meaningful? The instructor can guide the students toward a more nuanced understanding; reality and the engineer's analytical models can accurately and truly align, even as the understanding generated by their tools is not infinitely precise.

Modeling: Equivalent system of resultant force at a point and applied moment.

Students are now ready to create the first equivalent system. What does it mean to calculate the resultant force? The sum of the forces is equivalent to all the load on the system. In this case, students can write the mathematical expression for the resultant force (Equation 1) and simply stack the weights. To create an equivalent system with the force over the left support, the stack of weights can be placed over the left support. Do the scales now show the same reactions? If not, why not? What can be done to get the scales back to their original values? An applied moment could decrease the reaction on the right side and increase the reaction on the left side. Since an applied moment is a free vector, it can be applied anywhere and be externally equivalent. By twisting the frame and causing an applied moment, an equivalent system can be created. The physical reality modeled in Figure 6a and 7b is mathematically expressed in Equations 1 and 2 and idealized in Figure 6c. Instructors and students can continue the conversation concerning the true but idealized relationship between the mathematical expression, the mental model reflected in the sketch, and the physical reality.

$$\overrightarrow{F_R} = \sum \vec{F} = (-500\text{ lb} - 2000\text{ lb} - 2000\text{ lb})\,\hat{\jmath} = -4500\text{ lb}\,\hat{\jmath} \tag{1}$$

$$\overrightarrow{M_{R/A}} = \sum \vec{M} + \sum \overrightarrow{r_{/A}} \times \vec{F} = [0 - (2\text{ ft})(500\text{ lb}) - (12\text{ ft})(2000\text{ lb}) - (16\text{ ft})(2000\text{ lb})]$$
$$\overrightarrow{M_{R/A}} = -57000\text{ lb}\cdot\text{ft}\,\hat{k} \tag{2}$$

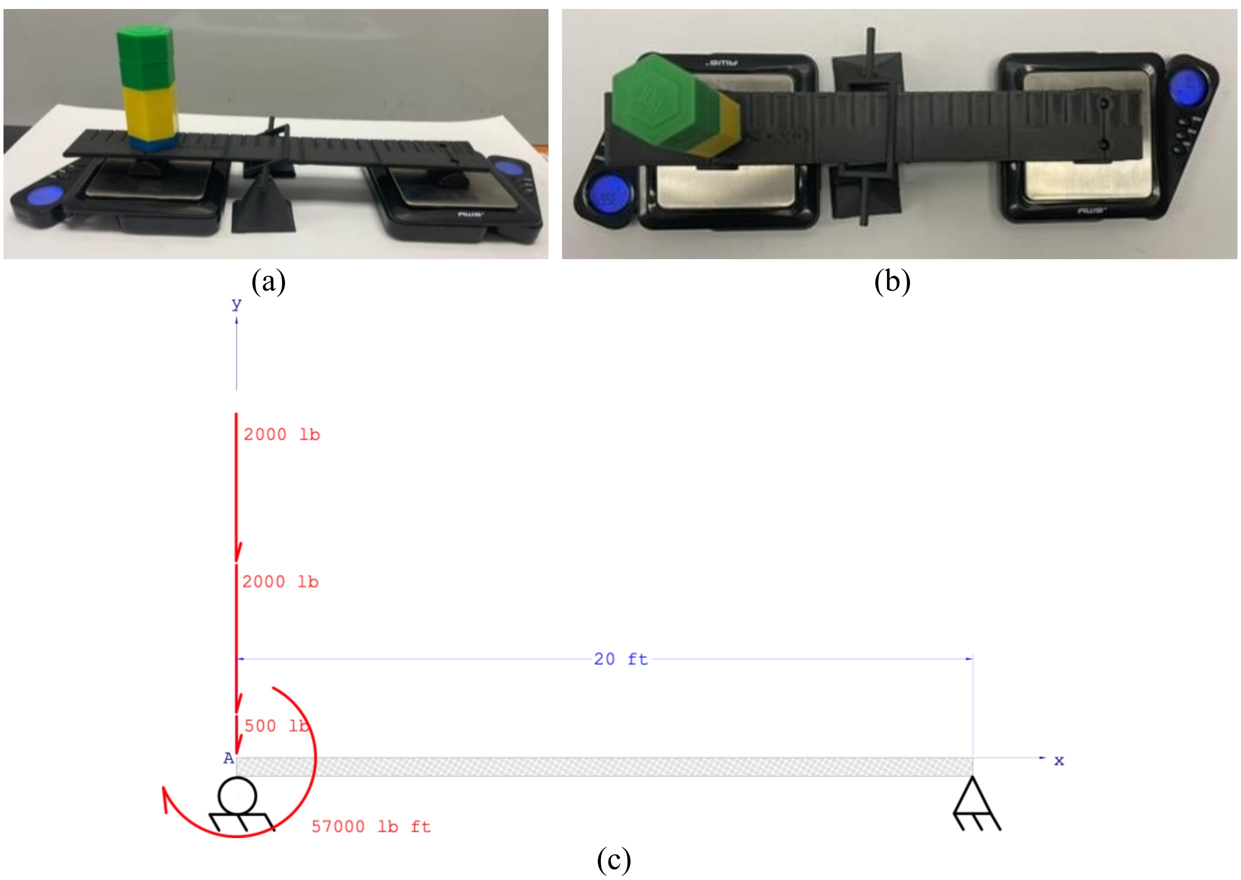


Figure 6. Simplified equivalent system example of a force over the left support and an applied moment: (a) profile view (b) plan view, (c) idealized schematic.

Modeling: Equivalent system of resultant force at a point

Students can now pursue the second simplified equivalent system. In this case, they are looking for the location for the force that will not require an applied moment to be equivalent. In other words, where would they place the resultant force (stack of weights) so that the reactions return to the original condition without needing an applied moment? The students slide the stack of weights back and forth on the beam, aiming to find the location that generates the same reactions from the complex loading. Again, groups of students might report their answers to the instructor. Again, the reality will have natural variation around an ideal approximated by the mean. Students can then use Equation 3.a to find the location. Since the horizontal force is zero, there is no meaningful value for y. Rather, students are only solving for the horizontal distance, x; therefore, in this case, Equation 3.a, simplifies to Equation 3.b. The final equivalent system is shown in Figure 7.

$$M_{R/A}\hat{k} = \left(xF_{Ry} - yF_{Rx}\right)\hat{k} \tag{3.a}$$

$$x = \frac{M_{R/A}}{F_{Ry}} = \frac{-570\ \text{k}\cdot\text{ft}}{-4.5\ \text{k}} = 12.67\ \text{ft} \tag{3.b}$$

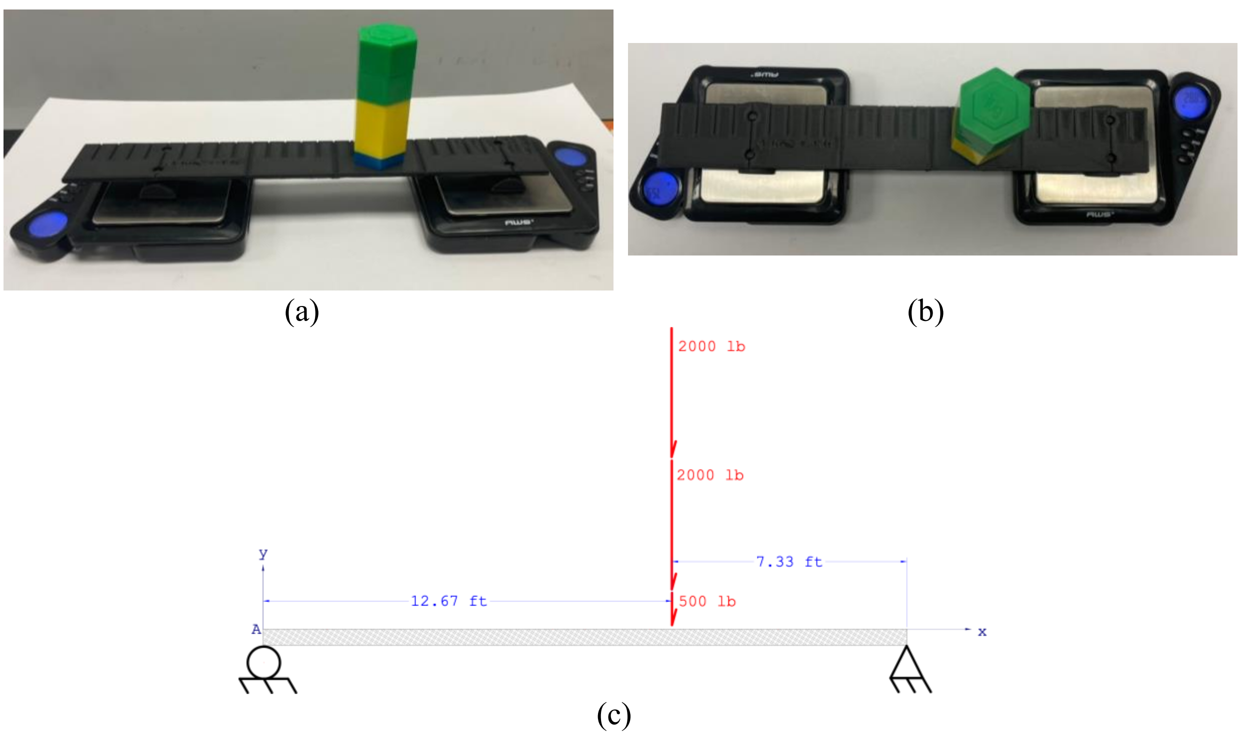


Figure 7. Simplified equivalent system example of a force at a point with no moment: (a) profile view (b) plan view, (c) idealized schematic.

If students have internalized the relationship between each system shown in Figures 4 to 7, their external equivalence, their identical reactions, and their identical resultants, they have mastered the content of the lesson. They should be ready to review and reinforce the content via homework problems. Additionally, the students are also primed to extend this knowledge into centers of gravity, center of mass and centroids in future class periods [12].

Distributed Loads

Main Idea

Distributed loads bring together the concepts of equivalent systems and centroids in a single topic. The topic sequence followed by the authors works from equivalent systems to centroids and closes the loop with equivalent systems involving distributed loads. By the end of this demonstration, students should be able to:

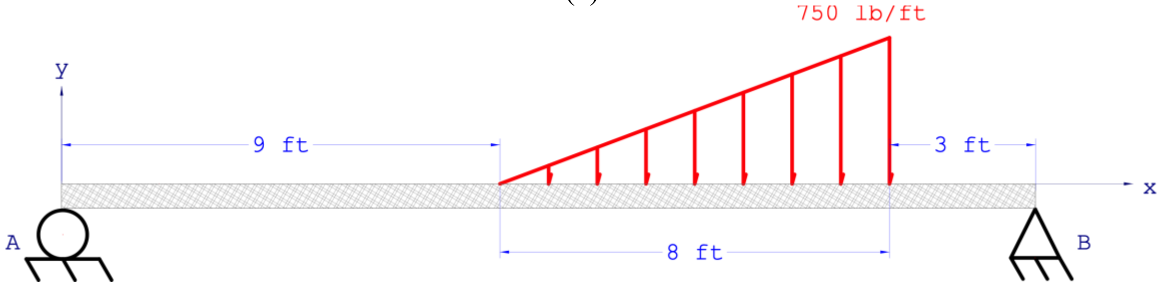
- Describe a distributed force.
- Calculate the resultant force and location for a distributed load.
- For distributed force, force, and moment loads, calculate an equivalent system of…
    - The resultant force and moment at a point.
    - The resultant force at a point.

Demonstration

Context: Risers on a Simply Supported Beam

Student engineers have been asked to help with the analysis of a floor beam supporting loaded choir risers as seen in Figure 8.a. The professional engineers have used the floor geometry, riser self-weight, and building codes to develop the idealized lived load pressure show in Figure 8.b. The student engineers are tasked with developing two equivalent systems: 1) resultant force at a point, and 2) the resultant force and moment at A.

(a)



(b)

Figure 8. Choir riser loads on a beam: (a) example risers [32] and (b) idealized loading developed by professional engineer.

Modeling: Distributed Load

Properly modeling a triangular distributed load using the gram weights is tricky. Student are asked to calculate the "area" of the triangle shown in Figure 8b given a base of 8 ft and a "height" of 750 lb/ft as seen in Equation 4. Using the area of a triangle and including units, the "area" is equal to 3000 lb. Students are then asked to model the distributed load using two blue 5 g weights and two green 10 g weights as seen in Figure 9. Student groups zero the scales with just the beams and supports, build the model, and report out the reactions measured by the scales. The instructor can again lead a discussion about accuracy, precision, the complexity of reality, and the usefulness of simplified models.

$$F_y \uparrow_y = -\tfrac{1}{2}(8\text{ ft})\left(750\,{}^{\text{lb}}/_{\text{ft}}\right) = -3000\text{ lb} \uparrow_y \qquad (4)$$

(a) (b)

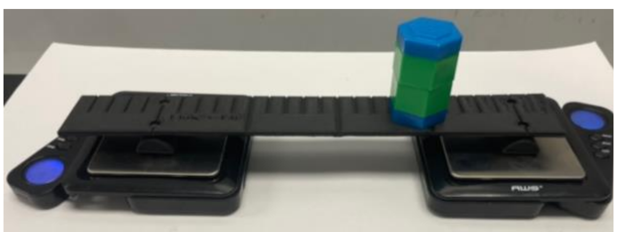
Figure 9. Loading set up for equivalent system example (a) profile view and (b) plan view.

Modeling: Equivalent system of force at a point

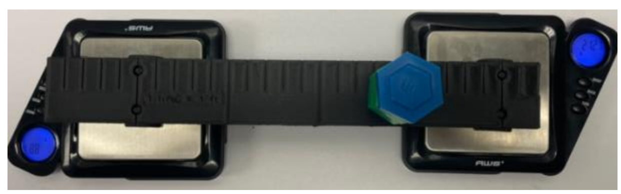
Next the students should recognize they have already calculated the resultant force by calculating the area of the triangle. Students can now model the resultant force by stacking the gram weights in a single column. They should then slide the column back and forth on the beam until they find a location where the reactions are the same as the complex loading. Again, student data can be collected, compared and discussed.

The instructor can then lead the students in finding the centroid of the triangle measured from point *A* or *B*. By referring to the geometry tables in the *FE Reference Handbook* [33] and using their knowledge of area centroids from previous lectures, students should use Equation 5 to generate an equivalent system like that shown in Figure 10. Special care must be taken to emphasize the offset adjustment required to measure the distance from point A and not from the tip or right angle ends of the triangle as described in the *FE Reference Handbook* geometry tables [33].

$$x \rightarrow^{x} = 9\text{ ft} + 8\text{ ft} - \frac{1}{3}(8\text{ ft}) = 14.33\text{ ft} \rightarrow^{x} \qquad (5)$$



(a)



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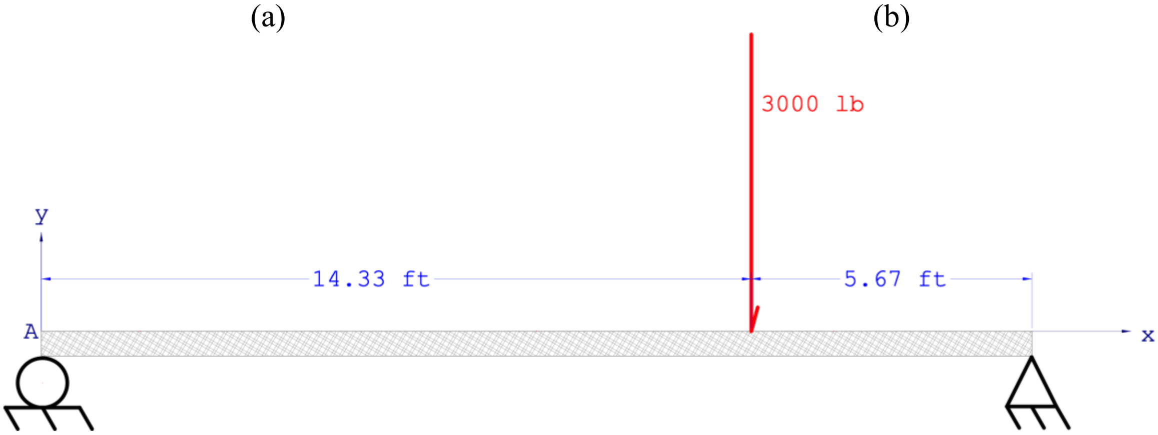


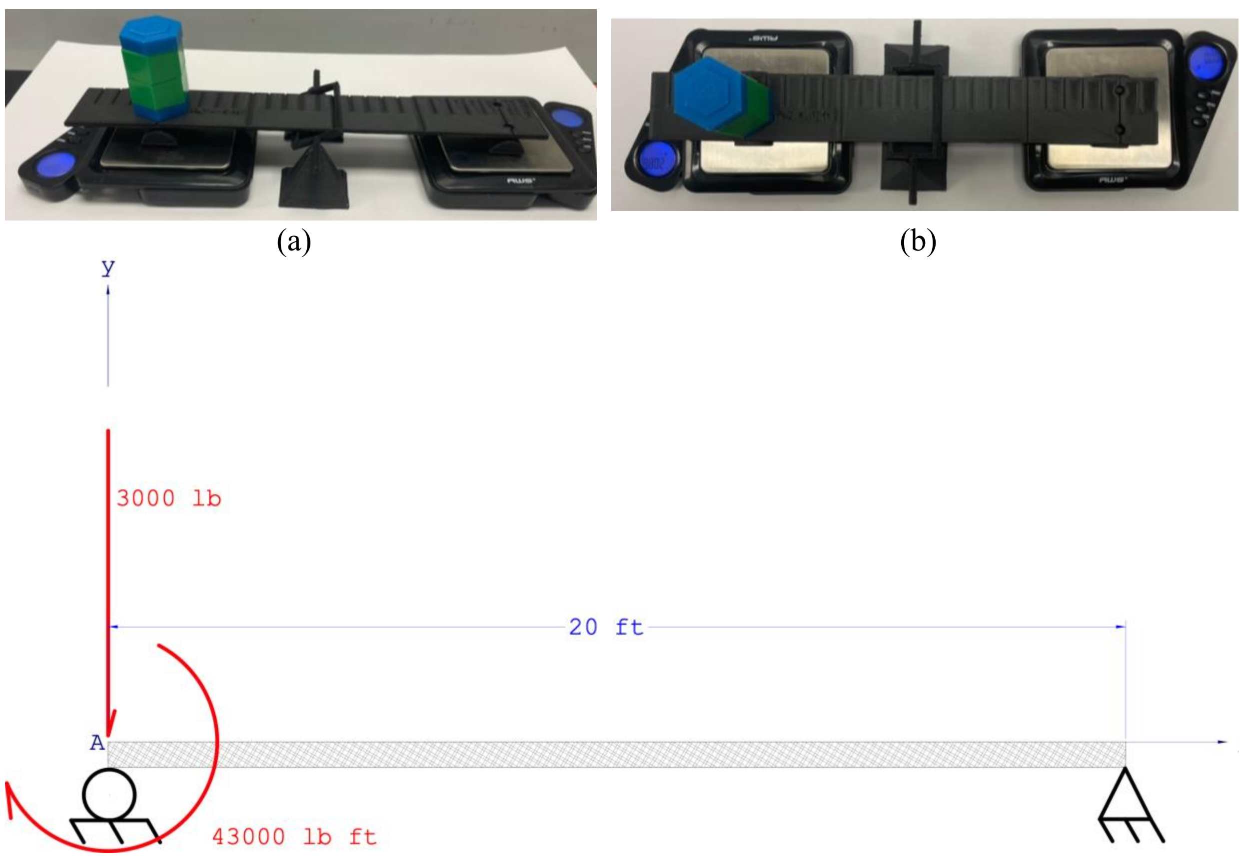
(c)

Figure 10. Simplified equivalent system example of a force at a point with no moment: (a) profile view (b) plan view, (c) idealized schematic.

Modeling: Equivalent system of force and moment at a point.

Finally, students can calculate the resultant moment about point *A* using the tools developed when discussing equivalent systems. See Equation 6. The model and idealized diagram for this resultant system is shown in Figure 11. The instructor can lead a discussion about the similarities between the distributed loads, centroids, and equivalent systems.

$$M_{/A} \uparrow\uparrow^{z} = -\left(9\text{ ft} + 8\text{ ft} - \tfrac{1}{3}(8\text{ ft})\right)\left(\tfrac{1}{2}(8\text{ ft})\left(750\,{}^{\text{lb}}/_{\text{ft}}\right)\right)\uparrow\uparrow^{z} = -43000\text{ lb}\cdot\text{ft}\uparrow\uparrow^{z} \qquad (6)$$



(c)

Figure 11. Simplified equivalent system example of a force over the left support and an applied moment: (a) profile view (b) plan view, (c) idealized schematic.

Students who can describe the relationships between the realities in Figures 8 to 11 in terms or equivalent systems, centroids, and distributed loads should be ready to practice and reinforce those concepts on distributed load homework problems. Students should now also have a comprehensive understanding of loads on beams expressed using vector mechanics to meaningfully engage with the calculation of reactions.

Reactions

Main Idea

Reactions on rigid bodies is the soul of the statics. This is the mandatory first step to any mechanics of materials problem and must be mastered by the student engineer. To that end, the beam model used to introduce equivalent systems returns. However, now students will engage with predicting the reactions that were merely observed in earlier exercises.

- Define…
  - A rigid body member (MBR).
  - Equations of equilibrium (EoE) for a force and moment system.
- For a single rigid body system in equilibrium…
  - Draw a free body diagram (FBD).
  - Write the EoE from the FBD.
  - Calculate force and moment reactions (RXNs) in 2D.

Demonstration

Context: The cantilevered floor system of the Kennedy Performing Arts Center

The student engineers have been asked to support the development a preforming arts hall like the Kennedy Performing Arts Center in Washington, DC. A prominent structural feature includes the cantilevered overhang that supports an outdoor space with trees above an active roadway. The white van in Figure 12 is about to drive underneath the overhang.

Figure 12. Kennedy Performing Art Center in Washington, DC [34].

The supervising engineer has simplified the loads on one of the beams supporting the overhang to those shown in Figure 13. The distributed load models the terraced seating inside the building the point load models the weight of the tree on the overhang. The students have been asked to calculate the reactions at the roller at point *A* and the pin at point *B*.

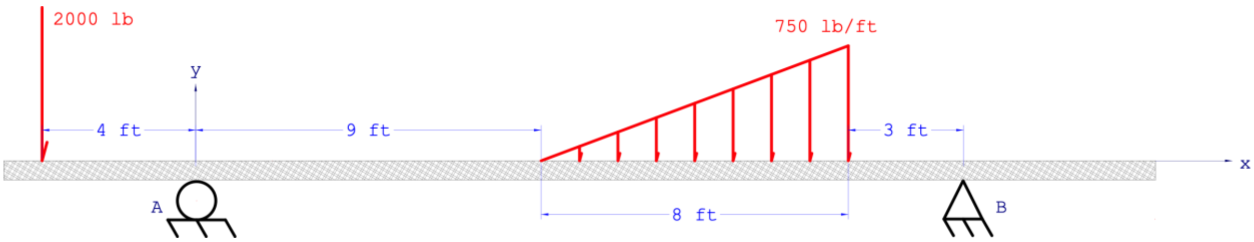


Figure 13. Beam load and support sketch.

Modeling: Distributed Load and Point Load

To introduce the idea of solving for reactions, the beam model makes its third appearance in the classroom. By now, students are familiar with the setup and what information the model can provide. Students can now set up the beam model leveraging their experience from earlier classes as seen in Figure 14. The distributed load is modeled just as it was in the previous example. The 2000 lb point load is applied using the 20 g yellow weight on the overhang.

(a) (b)

Figure 14. Actual loading set up for the reaction example (a) profile view and (b) plan view.

From the schematic and beam model, students work together to create a free body diagram for the beam. Students draw the body, external applied loads, axis, dimension and labels by copying the schematic drawing. The new concept comes with evaluating the roller and pin reactions at *A* and *B*. Student are encouraged to consider each axis and whether the support will resist translation or rotation along or about that axis. Where the answer is yes, they apply a labeled reaction force and assume the global positive direction. The roller at A only resists translation in the vertical direction, so the support is replaced with an upward reaction force. The pin at *B* resists translation in both the horizontal and vertical directions; therefore, the support is replaced with a horizontal reaction force pointed to the right and a vertical reaction force pointed up. Neither reaction resists rotation on its own, so the free body diagram does not need any moment reactions. Figure 15 shows the complete free body diagram for the example beam. Faculty can lead a discussion on whether the schematic and free body diagram are reasonable expressions of the beam model supports.

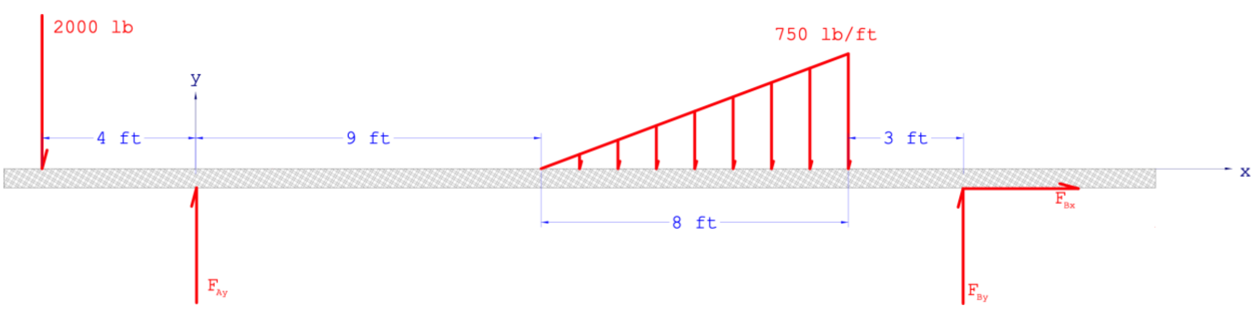


Figure 15. Free body diagram for reaction example beam model

Modeling: Finding Reactions for Simply-Supported Beams

With the free body diagram drawn, students should consider the as-of-yet unknown reactions. Can any students by observation estimate the horizontal reaction at *B*? Will the vertical reaction at *A* be larger or smaller than the reaction at *B*? Students can find some of the answers by reporting the vertical reactions measured by the kitchen scales. As different groups provide slightly different numbers, the instructor can again lead a discussion about the levels of precision and accuracy they can reasonably expect between reality, physical models, mental models, and analytical models.

Next, the students should write the equations of equilibrium. The authors prefer to start with the notation form captured in Equations C.9 and C.10 (see Appendix C) as seen in Equations 7 to 9 below. Working from left to right, the students should be able to see each force, distributed load and moment reflected in the equations of equilibrium. A series of questions can help students accurately transform the free body diagram into the equations of equilibrium.

Sum of the forces in the horizontal direction equals zero. Nothing translates left or right.

- Does the 2000lb force point in the x direction? No.
- Does the reaction force at A point in the x direction? No.
- Does the distributed load point in the x direction? No.
- Does the vertical reaction at B point in the x direction? No.
- Does the horizontal reaction B point in the x direction? Yes. In the positive or negative x direction? Positive. Therefore, include the positive force in Equation 7.

$$\sum F_x \rightarrow_x = 0 = F_{Bx} \qquad (7)$$

Sum of the forces in the vertical direction equals zero. Nothing translates up or down.

- Does the 2000 lb force point in the y direction? Yes. In the positive or negative y direction? Negative. Therefore, include a negative 2000 lb force in Equation 8.
- Does the reaction force at *A* point in the y direction? Yes. In the positive or negative y direction? Positive. Therefore, include a positive *Ay* force in Equation 8.
- Does the distributed load point in the y direction? Yes. In the positive or negative y direction? Negative. Therefore, include a negative calculation of the "area" of the distributed load in Equation 8.
- Does the vertical reaction at *B* point in the y direction? Yes. In the positive or negative y direction? Positive. Therefore, include a positive *By* force in Equation 8.
- Does the horizontal reaction *B* point in the x direction? No.

$$\sum F_y \uparrow_y = 0 = -(2000\ \text{lb}) + F_{Ay} - \tfrac{1}{2}(8\ \text{ft})\left(750\ {}^{\text{lb}}/_{\text{ft}}\right) + F_{By} \qquad (8)$$

Sum of the moments about point *A* equals zero. Nothing rotates about point *A*.

- Is there a perpendicular distance from point *A* for the 2000 lb force? Yes. Does the right-hand rule indicate counterclockwise (CCW) or clockwise (CW) rotation? CCW. Therefore, include a positive moment from the force in Equation 9.
- Is there a perpendicular distance from point *A* for the reaction force at A? No. Therefore, do *not* include a moment of from the reaction force *Ay* in Equation 9.

- Is there a perpendicular distance from point *A* for the centroid of the distributed load? Yes. Does the right-hand rule indicate counterclockwise (CCW) or clockwise (CW) rotation? CW. Therefore, include a negative moment of from the distributed load in Equation 9. Be sure to include the offset from point *A*.
- Is there a perpendicular distance from point *A* for the vertical reaction force at point *B*? Yes. Does the right-hand rule indicate counterclockwise (CCW) or clockwise (CW) rotation? CCW. Therefore, include a positive moment the reaction force *By* in Equation 9.
- Is there a perpendicular distance from point *A* for the horizontal reaction force at *B*? No. Therefore, do *not* include a moment of from the reaction force *Bx* in Equation 9.

$$\sum M_{/A} \uparrow\uparrow^{z} = 0 = (4\text{ ft})(2000\text{ lb}) - \left(9\text{ ft} + 8\text{ ft} - \tfrac{1}{3}(8\text{ ft})\right)\left(\tfrac{1}{2}(8\text{ ft})\left(750\,{}^{\text{lb}}/_{\text{ft}}\right)\right) + (20\text{ ft})F_{By} \qquad (9)$$

From these equations, the reaction forces can be calculated. Equation 7 states that the horizontal reaction at *B* is zero. Equation 9 can be solved to show that the vertical force at *B* is 1750 lb. By substituting this force into Equation 8 the vertical force at *A* resolves to 3250 lb. Students and instructors can compare these results to those measured by the scales. Why are they different or vary? Which should be used for further analysis and design? The instructor can also point out that the three written equations did require one substitution to solve the simultaneous equations. Alternatively, another moment equation could be written, summing about point *B*. This will not allow for the calculation of a new unknown, but it will make solving for the reaction force at *A* more direct. Alternatively, this equation can serve as an independent check on the substitution-based solution. Building Equation 10 also allows student to reinforce that the right-hand rule and rotation direction indicate the direction of the moment, not the direction of the force alone.

Sum of the moments about point *B* equals zero. Nothing rotates about point *B*.

- Is there a perpendicular distance from point *B* for the 2000 lb force? Yes. Does the right-hand rule indicate counterclockwise (CCW) or clockwise (CW) rotation? CCW. Therefore, include a positive moment from the force in Equation 10.
- Is there a perpendicular distance from point *B* for the vertical reaction force at point *A*? Yes. Does the right-hand rule indicate counterclockwise (CCW) or clockwise (CW) rotation? CW. Therefore, include a negative moment the reaction force *Ay* in Equation 10.
- Is there a perpendicular distance from point *B* for the centroid of the distributed load? Yes. Does the right-hand rule indicate counterclockwise (CCW) or clockwise (CW) rotation? CCW. Therefore, include a negative moment of from the distributed load in Equation 10. Be sure to include the offset from point *B*.
- Is there a perpendicular distance from point *B* for the reaction force at *B*? No. Therefore, do *not* include a moment of from the reaction force *By* in Equation 10.
- Is there a perpendicular distance from point *B* for the horizontal reaction force at *B*? No. Therefore, do *not* include a moment of from the reaction force *Bx* in Equation 10.

$$\sum M_{/B} \uparrow\uparrow^{z} = 0 = (24\text{ ft})(2000\text{ lb}) - (20\text{ ft})F_{Ay} + \left(3\text{ ft} + \tfrac{1}{3}(8\text{ ft})\right)\left(\tfrac{1}{2}(8\text{ ft})\left(750\,{}^{\text{lb}}/_{\text{ft}}\right)\right) \qquad (10)$$

Students who can describe the relationships between the realities in Figures 13 to 15 and Equations 7 to 10 should have mastery over the calculation of reactions for a simply supported beam. Students should now be ready to reinforce this mastery and expand it to other statical determinate systems on homework problems and projects.

Student Response

Student response to the beam model has been positive. Student engagement in the classroom has been high. Students seem to appreciate the real-world examples, the heavy emphasis on the relationships between reality, their mental models, and the analytical models and definitions discussed in class. Student performance on exams is reasonable with a consistent DFW rate for the course around 15%.

Equivalent Systems Assessment and Results

An equivalent systems problem (as shown in Figure 16) provided data on the effectiveness of the beam model. This problem contains a gear attached to a lever arm supporting forces at various angles. The problem prompt asked students to determine the magnitude, location, and direction of a single equivalent force and equivalent moment at Point A from the forces shown on the lever arm.

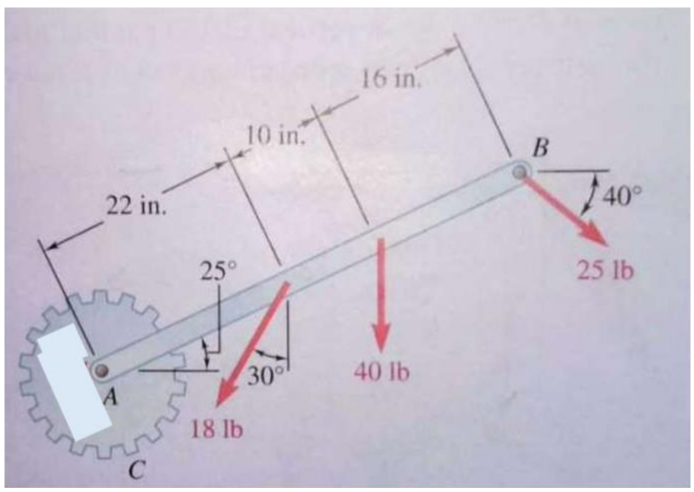


Figure 16. Equivalent systems assignment problem [35].

This assignment was given to statics students at three different universities: Miami University, University of Cincinnati and The Citadel. The statics students at The Citadel were the only students exposed to the beam model, while all other aspects of the statics courses were held similar between universities. The number of students for each class were as follows: Miami University had 34, University of Cincinnati had 43, and The Citadel had 45. The assignment was given to students at Miami University and University of Cincinnati during the fall of 2023, and to the students at The Citadel during the fall of 2024 and spring of 2025.

The same individual graded the assignment using the same rubric and process for each cohort, effectively eliminating grading variation. The results of this assignment are shown in Figure 17. The Citadel students outperformed the other two classes with an average percentage of 90.6%, while Miami University and University of Cincinnati resulted in average percentages of 85.4% and 88.7%. While the median grades did not follow the order of the averages, the lowest scores increased from 40% for students with no exposure to the beam model to 70% with the beam model. The beam model may be a useful tool for increasing understanding and mastery in the weakest students. No attempt was made to evaluate statistical significance given the small sample size and variations in the student cohorts. However, these results suggest the beam model demonstrations deliver positive benefit to students actively learning statics.

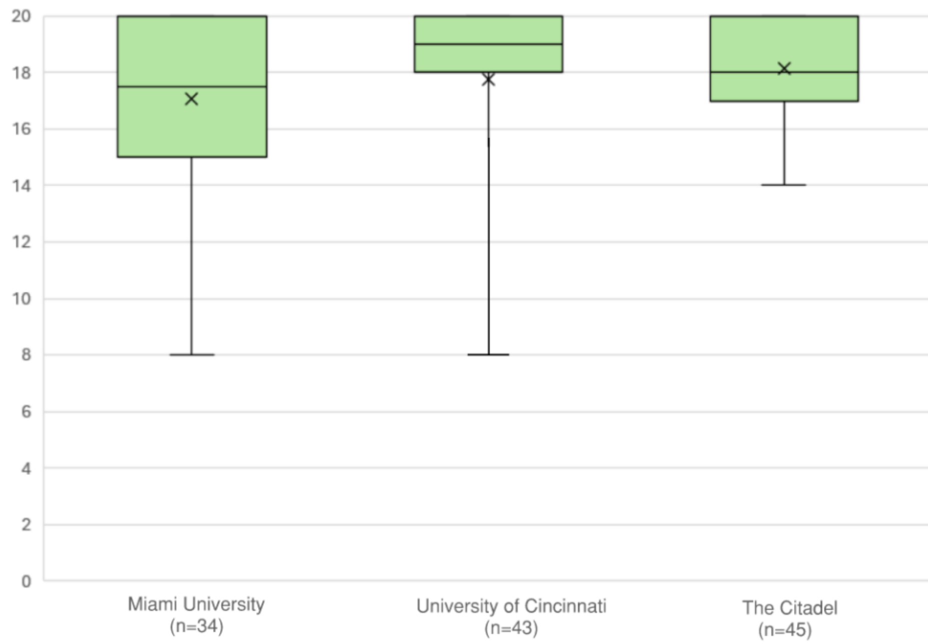


Figure 17. Average assignment grade by student cohort.

Further Work

Further work might be developed to apply the beam model to teach additional concepts. As noted by others, the beam model could be used to introduce the ideas of superposition [4]. Additionally, the beam model could easily make an appearance in later mechanics of materials and structural analysis courses. The beam model could provide helpful and contextualized review and encourage greater retention and knowledge transfer between courses.

Conclusion

The 3D printed beam model and associated demonstration materials have been an excellent and reoccurring feature of the authors' classrooms. Students have benefitted from exploring the concepts of equivalent systems, distributed loads, and reactions with an authoritative guide, clear real-world examples, and reinforcing modeling and analysis. Any new faculty seeking to teach this topic should find the content comprehensive enough to help students master the content and avoid common mistakes. Investing in Gregory's *7 Laws of Teaching* pedagogy allows faculty to rapidly develop strong actively learning, questioning, and discussion-based classroom experiences with less risk to student rapport than fully flipped classrooms or other more experimental teaching techniques. The use of the beam model engages even the weakest engineering students in curious investigation, the construction of strong mental models, and fluency in the analytical methods used by all engineers.

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