The Future of Engineering Education
2024 Annual Conference & Exposition
Oregon Convention Center
Portland, OR . June 23 - 26, 2024
ASEE
Paper ID #44202

Impact of Graphical Reasoning in Elementary Vector Analysis: A Case Study from Statics

Dr. Christopher Papadopoulos, University of Puerto Rico, Mayaguez Campus

Christopher Papadopoulos is Professor in the Department of Engineering Sciences and Materials at the University of Puerto Rico, Mayaguez Campus (UPRM). He earned B.S. degrees in Civil Engineering and Mathematics from Carnegie Mellon University (1993) and a PhD in Theoretical and Applied Mechanics from Cornell University (1999). He is active in the Mechanics Division and is also active in the area of Sustainability Engineering, including membership in the Engineering for One Planet Network.

Dr. Jean Carlos Batista Abreu, Elizabethtown College

Jean Batista Abreu earned his Ph.D. and M.S.E. at the Johns Hopkins University, M.S. at the University of Puerto Rico, and B.S.E. with Honors at the Pontificia Universidad Católica Madre y Maestra, all in Civil Engineering. Prior to joining Elizabethtown College, he served as a Visiting Assistant Professor of Mechanical and Civil Engineering at Bucknell University, and worked as a Structural Engineer at Acero Estrella.

Impact of Graphical Reasoning in Elementary Vector Analysis: A Case Study from Statics

Abstract

Engineering Statics is a fundamental engineering science course taken by many, if not most, engineering students. A basic topic introduced in Statics is the addition of vectors (also referred to as vector resultants). Typically, textbook and exam questions on this topic are algebraic in nature, with less attention given to graphical interpretation and representation. The authors of this study are interested in investigating the relationship between students' algebraic and graphical reasoning skills. The following questions have been posed and are being studied:

R1. How much does mastery of graphical analysis enhance student learning in Statics?
R2. Do students adopt the habit of redrawing generic figures to scale when given particular parameter values?

To answer these questions, the authors draw on their common approach to testing students on this and related topics. In particular, test questions on vector addition require students to perform both the algebraic calculations as well as a corresponding sketch, to good scale and proportion. Results demonstrate that, in general, a moderate number of students who attain the correct calculated values are unable to achieve an accurate corresponding diagram. As is true in many contexts of concept-based learning, this suggests that numerical proficiency is not sufficient to guarantee conceptual mastery.

The article explores reasons for this possible divergence and shares the instructors' perspectives and observations about how graphical analysis enhances student learning. In addition, the instructors present survey results on how students perceive the effectiveness of graphical methods in their learning of vector analysis.

1. Introduction

Essentially all presentations of vector operations in engineering courses are accompanied by "head to tail" diagrams that illustrate the concept of vector addition or resultants. The resulting laws of vector addition are then shown - visually - to follow the laws of triangular geometry and trigonometry. While most, if not all, textbooks provide geometrically accurate figures (i.e., the illustrations depict vectors with accurately measured lengths and angles) most treatments emphasize how to express vectors algebraically and to perform corresponding calculations. It is therefore possible that many students can complete a course such as Statics by performing operations but without ever drawing corresponding diagrams, or, in the cases when students do draw diagrams, it is unlikely that they draw them to accurate geometrical scale unless explicitly prompted.

This raises questions such as if creation and interpretation of accurate figures is a necessary part of understanding vector operations, and if such skills enhance, or at least correlate with, overall problem-solving performance. One approach to introduce graphical reasoning is via concept questions, in which students can identify from a given set of options which diagram(s) accurately represent a vector resultant or other characteristic. Another approach, as is explored in this article, is to introduce sketching as part of an expository problem-solving process in activities, assignments, and tests.

2. Literature Review

Little literature appears to be available that directly addresses the specific topic of drawing vectors to correct proportion in a mechanics class. However, several studies provided useful findings from similar or analogous contexts that can be extrapolated to our work.

In a study of physics students solving elementary problems of electrical field strength due to point charges, Maries & Singh [1] found that students who are not provided with a figure, yet who are instructed to draw one, are more likely to draw a useful 'expert diagram' than students who are simply provided with a generic schematic diagram that is not fully detailed. This possibly suggests when provided with a figure, students are willing to accept it as sufficient, and are not inclined to elaborate on it further, even when they have the skills to do so. Maries & Singh also discovered that students who drew more detailed diagrams performed better on the analytical results.

Flores et. al [2] measured the impact of teaching and emphasizing graphical manipulation on students' understanding of vectors in introductory physics courses. They found that, after traditional instruction (without emphasis on graphical interpretation), about 66% of the students assessed were unable to answer qualitative questions involving vector addition. Many of those students also exhibited difficulties when completing vector operations and interpreting results. After implementing instruction on graphical analysis, 60-90% of the students successfully determined vector magnitudes and directions.

In the context of vector fields, Hahn and Klein [3] utilized mobile and remote eye movement tracking data to assess the impact of drawing activities on students' performance and cognitive load. They found that students instructed with drawing activities generally pay more attention to essential aspects of the instruction and exhibit effective expert-like behavior in problem solving.

3. Methods

This is a collaborative work between the two authors, who use similar assessment tools and regularly compare notes on teaching Statics. Instructor A (Papadopoulos) teaches at the University of Puerto Rico, Mayagüez, a bilingual, public institution. Instructor B (Batista Abreu) teaches at Elizabethtown College, a small private institution. The study is based on analyzing results from a test question on vector resultants during the Fall 2023 semester, whose basic form is as follows:

> A diagram is provided showing two or three vectors, with certain parameters (magnitudes and angles/slopes relative to a set of reference axes) symbolically labeled. A written preamble then specifies the values of the illustrated parameters, but in a manner such that their values do not necessarily correspond to the apparent proportions indicated in the figure. The instructions are then as follows:

"(1) Determine the resultant vector, its magnitude, and its angle with respect to the x-axis; (2) Draw the corresponding vector polygon that illustrates how the given vectors sum to the resultant, …

… to approximate scale and proportion", in the case of Instructor A.
… accurately, using a ruler and protractor", in the case of Instructor B.

Although the tests are similar for both cohorts, Instructor A gave the test once, whereas Instructor B, using a mastery-based method, offered several opportunities for students to retake the test until they demonstrated complete mastery. The total experimental group consists of 91 students (Table 1).

Table 1. Cohort Information.

Instructor	**Female**	**Male**	**Total**
A	18	28	46
B	5	40	45
Total	23	68	91

4. Results and Discussion

The results are analyzed first by separating the students who attained the correct numerical results (C) from those with an error (E). Then, each of these sets is subdivided into categories of correct (C) vs incorrect (E) vector polygons. In the case of Instructor A, each of these is further subdivided into categories of drawing to appropriate scale (C) or not (E); this final subdivision was not done in the case of Instructor B because their evaluations on the diagrams combined the qualitative concept with the scaling.

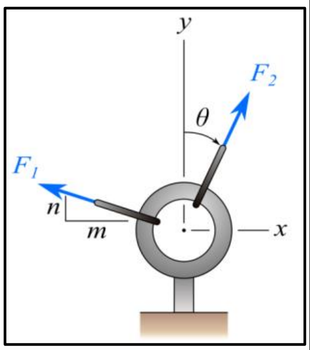
Examples (Instructor A)

In this test problem, students were provided with two vectors with magnitudes F_1 = 200 lbs and F_2 = 300 lbs. The orientation of F_1 is prescribed in slope format such that the horizontal component is 3/5 of the vertical component, even though the figure is (deliberately) drawn otherwise. Similarly, the orientation of F_2 is given by an angle of 57^o clockwise from the y-axis, even though the figure (deliberately) depicts this angle to be less than 45^o. To qualify for "good scale and proportion", Instructor A's expectations were that students should carefully examine the data and redraw the vectors to appropriate orientations at least to the closest octant, i.e., to correctly show the relative magnitudes of the Cartesian components of each vector. Thus, F_1 should be redrawn to illustrate greater vertical magnitude, and F_2 should be redrawn to show greater horizontal magnitude. Vector polygons that are not properly proportioned can still be qualitatively correct, if they show the proper sense of each vector and the proper head to tail addition. The results from evaluating the students' responses are reported in Table 2.

Table 2. Evaluations of Vector Resultant Question from Instructor A.

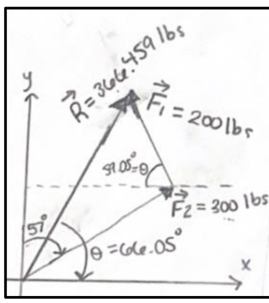
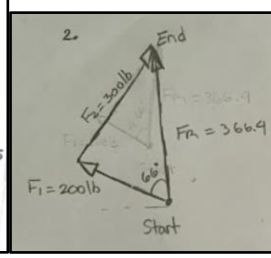
Calculations	**Vector Polygon**	**Itemized Results**
Correct 17/46 (37.0%)	Qualitatively correct with good scale (C,C,C) Qualitatively correct with poor scale (C,C,E) Qualitative errors but with good scale (C,E,C) Qualitative errors with poor scale (C,E,E)	4/46 (8.7%) 6/46 (13.0%) 3/46 (6.5%) 4/46 (8.7%)
Incorrect 29/46 (63.0%)	Qualitatively correct with good scale (E,C,C) Qualitatively correct with poor scale (E,C,E) Qualitative errors but with good scale (E,E,C) Qualitative errors and poor scale (E,E,E) Insufficiently developed to evaluate (n/a)	4/46 (8.7%) 5/46 (10.9%) 0/46 (0.0%) 17/46 (37.0%) 3/46 (6.5%)

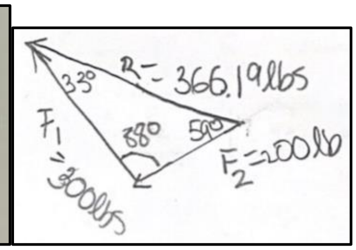
Figure 1 provides some excerpts of student work to illustrate several of the coded categories defined in Table 2.



Given:

F_1 = 200 lbs,
F_2 = 300 lbs,
θ = 57°, *m* = 3, *n* = 5

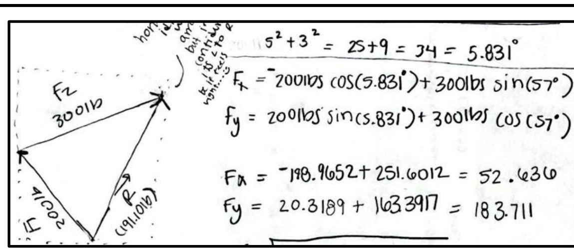





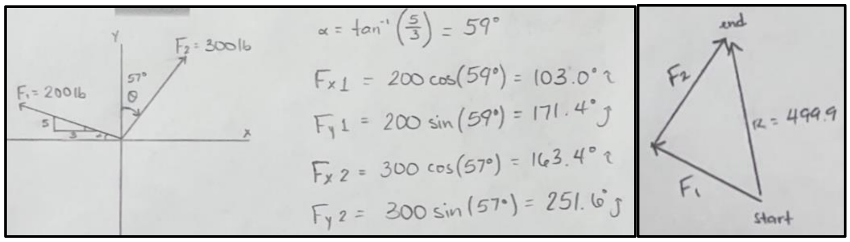
(C,C,C) (C,C,E) (C,E,C)

Notice that in the (C,C,E) case, the student did not interpret the resultant angle correctly, and/or was unable to reconcile the resultant's angle with the incorrect orientations of the vectors from their initial depiction.

In the (C,E,C) case, there is an error with the direction of F_2, but otherwise, a reasonable attempt was made to draw the vectors to good scale and proportion.



(E,C,C): Even though the student committed a calculation error (the result of the Pythagorean calculation at the top was erroneously interpreted as an angle), a reasonable attempt was made to draw the given input vectors F_1

<table>
<tr><td></td><td>and F_2 to good scale and proportion.


(E,C,E): Two calculation errors were made: the sign for F_{1x} should be negative, and the cos() and sin() terms are interchanged for the components of F_2. The diagram did not show the input vectors F_1 and F_2 to good scale and proportion (they were depicted similarly to the initial figure).</td></tr>
</table>

Figure 1. Examples of Student Work, Illustrating various Categories.

The results indicate that only a small fraction of students achieve an essentially perfect result in both the calculations and graphical analysis (4/46, 8.7%). Of the 17 students with the correct calculations, 10/17 (58.8%) did not use appropriate sense of proportion. This seems to corroborate the results of Maries & Singh, suggesting that students do not willfully provide additional details without prompting, even when they have a potential material effect on visualizing the problem. Of the 29 students who did not perform the correct calculations, 9/29 (31.0%) achieved a qualitatively accurate sketch, with nearly half of these using good scaling. This gives credence to the idea that some students do possess understanding that is not well measured by the analytical approach, and justifies the use of graphical reasoning as part of the problem-solving process.

To promote graphical reasoning in class, as well as other active learning techniques, Instructor A has developed a problem-based and cooperative learning model in the classroom that features weekly hands-on activities [4]. The very first activity on vector resultants requires scaled drawings of vector forces and their resultants that correspond to the measurements taken from a table-top experiment with spring scales. The majority of the subsequent activities also require drawing vectors and force polygons to scale.

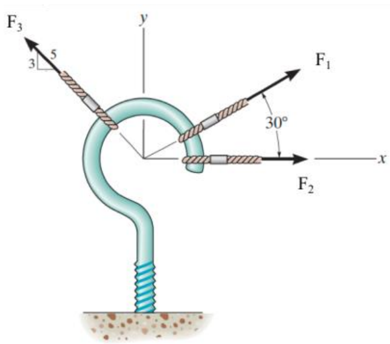
Examples (Instructor B)

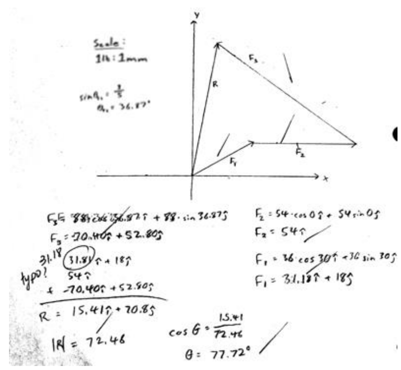
The Statics course offered weekly 30-minute long tests on the "*Force Vector Module*". Each test included two problems. The first problem presented a three-dimensional system with one related question while the second problem presented a two-dimensional system of three forces with two related questions. To solve the first problem, students needed to express position, unit and force vectors using Cartesian Vector Notation, add vectors, calculate magnitudes and/or apply the dot product. To solve the second problem (see sample problem in Figure 2), students needed to resolve vectors into rectangular components using angles or slopes, and determine resultants and their orientations with respect to a reference axis. In the second problem, the first question required numerical computations while the second question asked students to draw a force polygon to scale. The results in Tables 3 and 4 focus on the students' performance on the second problem (both questions). Table 3 only considers the students' performance on the first test offered in Fall 2023. Table 4 considers all the attempted tests given throughout the Fall 2023 semester in the mastery-based Statics course.

Consider the three forces acting on the eye hook. Assume F_1 = 47 lb, F_2 = 46 lb, and F_3 = 84 lb.

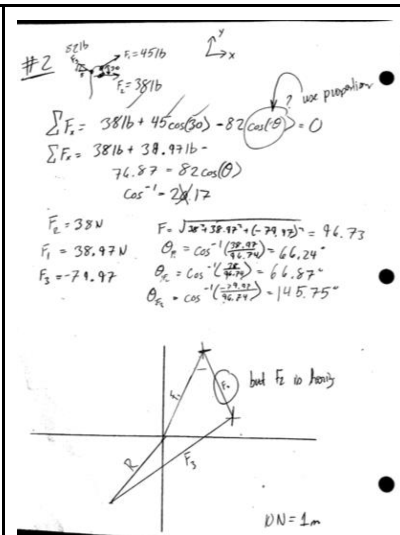
Determine the angle formed by the resultant force and the x-axis.

Draw the forces polygon to scale, including the original forces and the resultant.

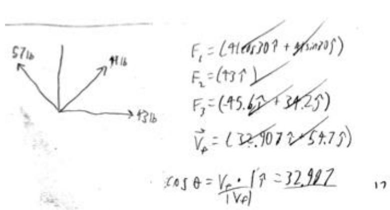




(C,C): The student provided and used a consistent scale and the angles were accurately measured. Forces are labeled. Graphical results validate numerical answers.



(E,E): The student did not interpret the given information correctly which led to calculation errors. Also the orientations of the force vectors are incorrect.



(C,E): The student correctly calculated an angle formed by the resultant force and the horizontal axis, but was unable to graphically represent the resultant force.

[There were no examples in this category. All the solutions that included correct drawings also included correct calculations.]

(E,C)

Figure 2. Examples of student work from Instructor B.

Table 3. Instructor B - Results include submissions on the 1st attempted test in Fall 2023.

Calculations	**Vector Polygon**	**Itemized Results**
Correct 20/36 (55.6%)	Qualitatively correct with good scale (C,C) Qualitative errors and/or poor scale (C,E)	11/36 (30.6%) 9/36 (25.0%)
Incorrect 16/36 (44.4%)	Qualitatively correct with good scale (E,C) Qualitative errors and/or poor scale (E,E)	0/36 (0.0%) 16/36 (44.4%)

Table 4. Instructor B - Results include all the submissions in Fall 2023.

Calculations	**Vector Polygon**	**Itemized Results**
Correct 68/119 (57.1%)	Qualitatively correct with good scale (C,C) Qualitative errors and/or poor scale (C,E)	45/119 (37.8%) 23/119 (19.3%)
Incorrect 51/119 (42.8%)	Qualitatively correct with good scale (E,C) Qualitative errors and/or poor scale (E,E)	10/119 (8.4%) 41/119 (34.4%)

The results in Table 3 show that by the second week of classes (when the first test was offered) all the students who had mastered the graphical approach also mastered the numerical approach.

Based on Table 4, a significant percentage of solutions include either correct calculations and correct polygons (37.8%) or incorrect calculations and incorrect polygons (34.4%). When drawing polygons, typical errors include inconsistent scales, inaccurate angle measurements, reversed vector senses, and incomplete polygons (i.e., missing one or more vectors). Interestingly, the number of solutions with correct vector polygons and incorrect calculations (8.4%) is small compared to the number of solutions with incorrect vector polygons and correct calculations (19.3%). The results suggest that students are more likely to complete the numerical work correctly once they develop graphical reasoning (45/55, 81.8%). In contrast, a relatively small number of students demonstrate analytical proficiency without demonstrating graphical reasoning (23/64, 35.9%). Some of these students seemed to memorize a procedure to obtain numerical solutions without understanding basic vector operations and successfully interpreting results.

Student Perception (Instructors A and B)

In a voluntary post-course survey, students were asked the following open-ended questions to assess their perception about the importance of graphical analysis and the use of scale.

> Q1. Does drawing the force polygon help you to understand vector operations? Why or why not, and to what degree?
> Q2. Is it important to draw vectors to proper scale and proportion? Why or why not?

Most students perceive that drawing force polygons is important and that it helps them understand vector operations because it allows them to “visualize the applied forces and how they are in equilibrium” (Table 5). In their responses, students acknowledged that using a proper scale helps them better understand the problem statement, comprehend Static equilibrium, and validate their calculations and results. Some students believe these advantages are particularly relevant “for beginners” and less crucial once they grasp the concepts. A few students found the drawings irrelevant given that they can mathematically understand the concepts.

Table 5. Post-course survey results.

Instructor (sample size)	**Question**	**Answer**		
		Yes	**Moderately**	**No**
A (N = 8)	Q1	100.0%	0.0%	0.0%
B (N = 19)		63.2%	21.1%	15.8%
A (N = 8)	Q2	87.5%	12.5%	0.0%
B (N = 19)		63.2%	15.8%	21.1%

5. Students Performance on Conceptual Questions

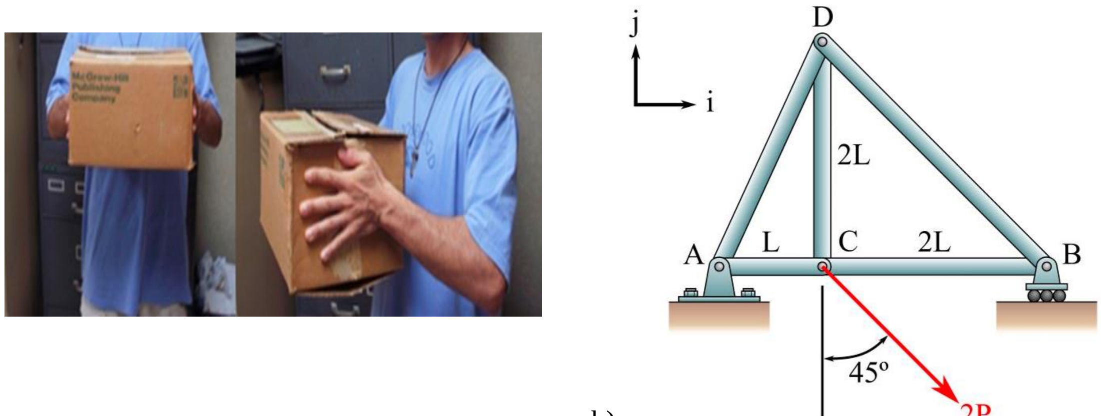
Instructor B assessed students' understanding of foundational vector operations by evaluating their responses to two conceptual questions that required visualizing the relative magnitude of forces and their directions. These questions were adopted from the AIChE Concept Warehouse [5]. The first problem (Figure 3-a) required students to consider a person holding a box of books with flat hands. Students were asked to explain what happens to the friction force applied by the hands onto the sides of the box if the hands press harder. Students were expected to conduct a graphical analysis on the system of vectors with known directions and unknown numerical magnitudes. The second question presented a planar truss system subjected to a concentrated force at a joint (Figure 3-b). Students were asked to identify the number of members in tension, members in compression, and (potentially) zero-force members. Students were expected to qualitatively analyze the truss by inspecting the joints and, therefore, determine the relative magnitude of force vectors and their directions. Both problems were presented as multiple-choice questions with follow-ups asking students to explain their reasoning.

The control group (N = 59, Fall 2021-2022) consisted of students who were never required to draw vector polygons in Statics. The experimental group (N = 45, Fall 2023) took Statics with the same instructor and, in addition to being exposed to the same content and expectations, was required to draw vector polygons in Statics tests. The intervention consisted of a 20-minute-long workshop during the first week of classes focused on drawing force polygons to scale. In this workshop, one step-by-step example was presented to and discussed with the students. Then, students were invited to complete a similar practice problem.

Students' responses to the two conceptual questions show that the experimental group (which intentionally developed graphical reasoning in vector analysis) demonstrated a better understanding of Statics concepts (Figure 4). Therefore, it seems possible to argue that graphical reasoning helps improve the overall understanding of vector analysis.

Another interesting fact about the control and experimental groups is that the average number of exam retakes in the "*Force Vector Module"* per student was 2 for both, even though the latter had the additional challenge of demonstrating the ability to draw force polygons to scale and interpret related results. Therefore, adding the extra requirement does not appear to impede nor delay student progress.

a)



b)

Figure 3. AIChE Concept Warehouse questions a) ID 4497 and b) ID 4756.

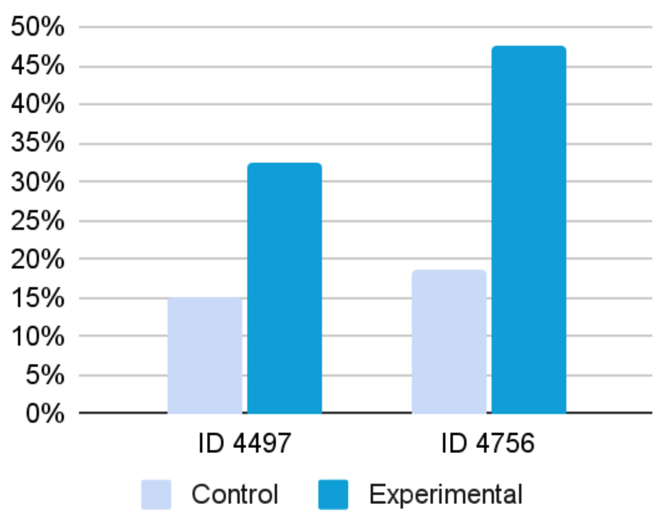


Figure 4. Percentage of students who submitted correct answers and reasoning.

6. Conclusions

The results of this work provide plausible evidence that graphical reasoning is justified as part of an introductory course in engineering mechanics. There is an appreciable number of students in both cohorts who demonstrate good graphical skills even in the absence of performing correct calculations, suggesting that graphical reasoning can reach students who are weaker in algebraic skills. There is further evidence, in the case of Instructor B, that the use of graphical methods translates into performance gains in later topics, as measured by results in concept questions from the Concept Warehouse. Finally, based on the survey results, students generally appear to acknowledge the usefulness of drawing careful figures.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1821445. The authors gratefully acknowledge this support.

References

[1] A. Maries and C. Singh, “To use or not to use diagrams: The effect of drawing a diagram in solving introductory physics problems,” presented at the 2012 Physics Education Research Conference, Philadelphia, PA, USA, 2013, pp. 282–285. doi: 10.1063/1.4789707.

[2] S. Flores, S. E. Kanim, and C. H. Kautz, “Student use of vectors in introductory mechanics,” *Am. J. Phys.*, vol. 72, no. 4, pp. 460–468, Apr. 2004, doi: 10.1119/1.1648686.

[3] L. Hahn and P. Klein, “Analysis of eye movements to study drawing in the context of vector fields,” *Front. Educ.*, vol. 8, p. 1162281, Jun. 2023, doi: 10.3389/feduc.2023.1162281.

[4] C. Papadopoulos, “Changing the Static: Insights and Early Results of a Shift toward a Studio-Style Statics Class,” American Society for Engineering Education Annual Conference & Exposition, Baltimore, MD, June 2023. doi: 10.18260/1-2--43170

[5] M. D. Koretsky, J. L. Falconer, B. J. Brooks, D. M. Gilbuena, D. L. Silverstein, C. Smith, and M. Miletic, “The AIChE Concept Warehouse: A Web-Based Tool to Promote Concept-Based Instruction,” Advances in Engineering Education, vol. 4, no. 1. 2014.