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# The Impact of Inquiry-Oriented, Differential-Equations Instruction on Students' Performance and Beliefs about Mathematics

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# The Impact of Inquiry-Oriented Differential Equations Instruction on Students' Performance and Beliefs about Mathematics

Inquiry-oriented (IO) instruction is one of many inductive teaching approaches that relies heavily on active student learning. However, there are key features that distinguish IO instruction from active learning in other classrooms. Traditionally, if students actively participate in a university mathematics class, it is after an instructor has presented key concepts and procedures. That is, their engagement is that of practice. In an IO classroom, however, students are expected to reinvent mathematics in their quest to solve real-world problems [1]. Therefore, the applications precede and motivate, rather than follow, the theory.

In an IO course, students are presented with novel problems; they are not able to mimic previously learned methods to solve the problems. Instead, they have to engage in the real practice of problem solving. They pose and test conjectures. They construct solution methods and arguments to justify those methods, and they present and evaluate their and others' arguments. This requires engaging in mathematical discussions with other students and the instructor [1]. As such, engagement in inquiry is just as important for the instructor as the students. As students engage in inquiry of mathematics, instructors inquire into student thinking and reasoning [2][3]. At times in an IO course, students will be "stuck" for significant periods of time. This is seen not as a failure, but as a normal part of the problem solving process.

**Inquiry-oriented Differential Equations**

Traditional instruction in differential equations has used a lecture format to present analytic methods for solving specific types of equations [4]. While efficient, this instructional approach presents challenges to students developing deep conceptual understanding of course topics and leaves students at a disadvantage when needing to apply course content in subsequent courses [5][6].

Inquiry-oriented differential equations (IODE), specifically, has been applied formally and researched for the past three decades. In addition to adhering to the principles described above, IODE applies theory developed from years of research specific to students' understanding of differential equations. For example, the literature shows that most often, students who have completed a course in differential equations do not demonstrate a consistent understanding of a solution being a *function* [7]. Therefore, activities in the IODE course consistently ask students to interpret their solutions. Additionally, while traditional courses focus predominantly on analytic methods, the IODE course balances analytic, graphical, and numerical methods [3]. Students are required not only to use all three, but are given opportunities to discover the connections between them, strengthening their conceptual understanding.

Data from the past two decades has shown that inquiry-oriented instruction leads to gains in conceptual knowledge without sacrificing procedural knowledge [8]. Additionally, some studies have shown that this type of instruction narrows the gap for underperforming students. For example, Laursen and colleagues showed that while college mathematics courses normally result in students' decreased mathematical confidence, and to a greater degree for women, this effect was diminished in an inquiry-oriented classrooms and the confidence level for women actually increased [9]. Moreover, proponents of IODE instruction argue that because of increased focus of conceptual understanding, retention of both conceptual and procedural knowledge increases for IODE students [10].

## Current Study and Research Questions

This study reports on implementing inquiry-oriented instruction in a course for engineering students that had been well received by students, but was not accomplishing some of the essential goals that instructors had for preparing students for the engineering curriculum. In most sections of the class, a typical class meeting consisted of a lecture in which the instructor introduced a type of differential equation and showed the students how to identify and solve that type of differential equation. The students then completed a worksheet that had them repeat the process that the professor just demonstrated on one or more examples of that type of differential equation. In short, students were trained to be good at mimicking a process and identifying when to use that process. Applications were briefly mentioned, but accounted for only a small portion of the class and were not integrated with the rest of the material. The design of the new course had four goals. First, it would motivate mathematical content with real world problems. Second, students would learn to not only solve but *construct* differential equations that model physical situations. Third, the course would connect analytic, numerical, and graphical representations and solution methods. Finally, students would utilize discussion and argumentation in problem solving to *discover* content that had traditionally been presented by an instructor. To assess the impact of the IODE curriculum, we pose the following research questions:

1. How does the IODE curriculum impact student performance?
2. How does the IODE curriculum impact students' views about mathematics?

## Methods

### *Context*

The first and second authors each taught two in-person sections of Ordinary Differential Equations in Fall 2019 and Fall 2021. The course was taught traditionally in Fall 2019 and fully implemented as an IODE course for the first time in Fall 2021. We began developing draft lesson plans in Fall 2019, starting with materials provided by NSF Project Award #1431641: Teaching Inquiry-Oriented Mathematics: Establishing Supports. The same semester, we held conversations with faculty who teach courses for which ODE is a prerequisite to identify content that is most essential, and who recommended problems and activities that are more specific to engineering. In Spring 2020, we held mock lessons with previous ODE students to implement and then modify drafts of lesson plans. While we had originally planned to do a full implementation of the IODE curriculum in Fall 2020, due to the pandemic and classes being online, we waited until Fall 2021, when classes resumed in person, to fully implement the new curriculum and collect data for comparison to Fall 2019.

In Fall of 2019, each class started with a lecture going over the topic for the day, followed by time in class for students to complete problems by mimicking examples from the instructor had just worked. Sometimes, a small portion of the lecture would be dedicated to why the given technique worked. However, the focus was on *what* to do rather than *why* it worked.

The IODE curriculum implemented in Fall 2021 covered largely the same topics as Fall 2019 (with homogeneous equations (solved using v=y/x), Bernoulli equations, Euler equations, using roots of unity to solve higher order equations with constant coefficients, variation of parameters for third order equations, linear systems of equations with three or more equations, Abel's theorem, variation of parameters for linear systems of equations and matching between the complementary solution and the proper form of the particular solution in undetermined coefficients for systems of equations being removed, and nonlinear systems of equations being added in). Each day students worked in small groups to complete inquiry-oriented assignments. Some of these assignments guided students to discover methods for solving differential equations by building on already existing knowledge. For example, the method of separation of variables was built through pre-existing knowledge of the chain rule. An example of a sequence of activities for first order linear equations solved with an integrating factor is given in the appendix.

In Fall 2021, connections were made often between symbolic, graphical, and numerical representations and students were required to use these collectively to predict and interpret solutions. For example, after constructing an equation to model the rate of change of temperature of a cup of coffee, students first produced a solution curve using the slope field. They used this to estimate the time required for the coffee to reach a specific temperature. They then estimated this time numerically using Euler's Method. Finally, they solved the equation using separation of variables and compared the results of the three methods. Accordingly, assessments in 2021 presented problems using multiple representations, where the 2019 assessments primarily used symbolic representations and solution methods.

The Fall 2021 curriculum focused on conceptual understanding in ways the 2019 curriculum did not. For example, we often asked students to create differential equations (as opposed to only solving them), to explain the relationship between variables in an equation and how one impacted the other, and to interpret what solutions of an equation with initial conditions implied about the short and long-term behavior of a quantity. We also guided students to discover theorems that had been presented as fact in the traditional curriculum. Other conceptual test questions from Fall of 2019 are given in the appendix In Fall 2019, instructors often showed derivations and provided explanations, but did not expect students to develop these and did not assess students' understanding, but rather their procedural fluency.

*Data Collection: Views about Mathematics*

At the beginning and end of the Fall 2019 and Fall 2021 semesters, the Views about Mathematics Survey (VAMS) was administered via Qualtrics during class. In Fall 2019, 150 students [96 males, 54 females] out of 173 students [110 males, 63 females] in four sections of APMA2130, Ordinary Differential Equations, completed the Views about Mathematics Survey. In Fall 2021, 192 students [123 males, 69 females] out of 236 students [152 males, 84 females] in five sections of APMA2130, completed the same survey.

This instrument has been used previously to assess the impact of IODE instruction [1]. Each VAMS item presents a statement and a binary choice and students express their preference for one of the choices on a scale of 1-7 [11][12]. For example, one statement students were asked to react to was:

For me, doing well in mathematics courses depends on:

a) How much effort I put into studying.
b) How well the teacher explains things in class.

Students were asked to react to this by choosing one of the following:

1. Only a
2. Mostly a
3. More a than b
4. Equally a and b
5. More b than a
6. Mostly b
7. Only b
8. Neither a nor b

The VAMS items have been administered to mathematicians to establish "expert" views for each item. Student responses are then scored, not for "correctness" but in comparison to the expert view. For the previous item, answers were grouped as follows:

- Expert View: Options 1-3
- Mixed View: Option 4
- Folk View: Options 5-7

For each student, the number of responses that matched the expert view and the number that matched the folk view were totaled. These totals were used to categorize each student as having an Expert, Upper Transitional, Lower Transitional, or Naïve view [14]. Each student had a classification at the beginning and end of the semester (since they completed the VAMS survey twice) and we were able to quantify each student's change in classification from the beginning to the end of the semester. We then compared these changes from Fall 2019 to Fall 2021.

*Data Collection: Performance*

To determine whether there was a change in students' procedural fluency and in conceptual understanding, we gave a set of 5 test questions to students in Fall of 2019 and again in Fall of 2021. This set of questions consisted of two conceptual questions, and three procedural ones. We classify questions as conceptual questions if they require students to apply ideas from the class in a new way, synthesize multiple concepts, or if they require students to explain their answers. On the other hand, a procedural question is one that requires students to apply the procedure for solving a problem (in this class, usually solving a differential equation) in a way that students have seen previously.

In 2022, these questions were re-scored by one person (they originally had different rubrics and graders) and the resulting scores were compared. The two conceptual problems are pictured below. In the first one, students needed to understand what it means for a function to be a solution of a differential equation to determine whether the given function solved the given differential equation. In the second one, they needed to understand the existence and uniqueness theorem for first order linear equations enough to determine its conclusions for two different initial value problems with the same differential equations, but different initial conditions.

**Verification question, Test 1:**

Is $f(t) = ln\,(t)/t$ a solution of $t^3y^3 + t^2y^2 + t^3ylnt \cdot y'(t) = 2(lnt)^2$?

Justify your answer by showing all necessary work.

**Existence and Uniqueness, Final Exam:**

Consider the **linear** IVP: $y'(t) + p(t)y(t) = g) \;\; y(t_0) = y_0$

If $p(t)$ and$g(t)$ are continuous on $(\alpha, \beta)$ containing $t_0$, then there exists a **unique** solution $y = f(t)$ on the entire interval $(\alpha, \beta)$.

Determine (without solving the equation) the largest interval on which the solution of each given initial value problem is guaranteed to exist.

Justify your answers.

(a) $(t - 3y' + ylnt = t^2, \quad \mathrm{y}(7) = 9$
(b) $(t - 3y' + ylnt = t^2, \quad \mathrm{y}(1) = -4$

The three procedural questions are given below. There were two questions that dealt with linear first order equations. The other question asked students to use the Laplace transform to solve an initial value problem whose forcing term included the unit impulse function.

**Linear Equation, Test 1:**

Solve the following initial value problem:

$$t^2y' + 4ty = t^3, \quad \mathrm{y}(1) = 3$$

**Linear Equation, Final Exam:**

Find the general solution to the differential equation:

$$y'(t) = 3y(t) = \frac{1}{2}e^{3t}$$

**Laplace IVP, Final Exam:**

Use Laplace transforms to solve the initial value problem below.

$$y''(t) + 6y'(t) + 25y(t) = \delta(t - 5), y(0) = 1, y'(0) = -3$$

Analysis: Performance

To statistically analyze results between the two semesters, we started by finding the average score on each of the five problems. Then we used a t-test for the difference between the two means for each question. We tested the hypothesis that there was no difference between the means.

**Results**

*Performance*

Using the null hypothesis that there is no difference between the means, we got the following results, where the t-values are calculated by subtracting the 2019 average scores from the 2021 average scores, and then dividing by the square root of the sum of the sample standard deviations squared divided by each sample size. Below, we've reported the mean and sample standard deviation in terms of percentages rather than in terms of the raw scores.

| | Conceptual | | | | Procedural | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Verify | | Exis. & Uniq. | | Linear Test 1 | | Linear Final | | Laplace IVP | |
| | df=9942 | | df=7660 | | df=6614 | | df=7325 | | df=8252 | |
| | F19<br>n=116 | F21<br>n=79 | F19<br>n=116 | F21<br>n=79 | F19<br>n=116 | F21<br>n=79 | F19<br>n=116 | F21<br>n=79 | F19<br>n=116 | F21<br>n=79 |
| Mean | 55.7% | 84.3% | 86.9% | 83.5% | 95.9% | 85.3% | 94.7% | 90.7% | 83.3% | 88.8% |
| SSD | 45.7% | 24.3% | 24.2% | 26.4% | 12.3% | 26.6% | 17.9% | 22.9% | 22.9% | 20.0% |
| t-value | 5.66 | | -0.89 | | -3.31 | | -1.31 | | 1.79 | |
| P-value | <0.001 | | 0.375 | | 0.001 | | 0.190 | | 0.074 | |

Using a 5% significance level with a Bonferroni correction, so that results are only significant at the 5%/7=0.714% level, we can see statistically significant results for the conceptual question asking students to determine whether a function solves a given differential equation, and the question on the first test asking students to solve a first-order linear initial value problem. The other questions do not show a significant difference on an individual basis.

We also calculated combined scores for the procedural questions and the conceptual ones. The results are given below.

| | Conceptual | | Procedural | |
|---|---|---|---|---|
| | df=7400 | | df=11241 | |
| | F19<br>n=116 | F21<br>n=79 | F19<br>n=116 | F21<br>n=79 |
| Mean | 74.9% | 83.8% | 90.3% | 88.2% |
| SSD | 25.9% | 21.3% | 13.2% | 16.2% |
| t-value | 2.64 | | -0.91 | |
| P-value | 0.008 | | 0.360 | |

Here, we can see that while the procedural average in F21 is slightly lower than it was in F19, this difference is not statistically significant. The difference between performance in the conceptual questions is also not statistically significant because 0.008<0.00714, but it is close. Without the Bonferroni correction, it would have been statistically significant.

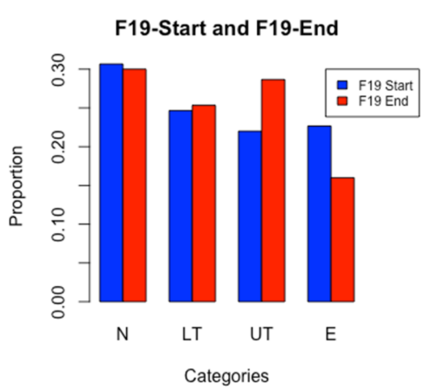
*Views about Mathematics*

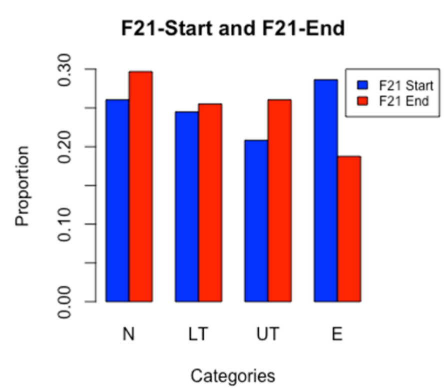
Here we show results of comparisons of students' views about mathematics (as measured by the VAMS survey) in Fall 2019 and Fall 2021. We also compare results for male and female students. Since observations by semesters, and, later, by gender, are not equal in size, we normalize all data to percentages. The values on the vertical axis will be between 0 and 1, and bar height will show the percentage of students in the corresponding profile in the given semester.

The first data we looked at is the distribution of the 4 categories, as described in VAMS:

- N: Naïve
- LT: Lower Transitional
- UT: Upper Transitional
- E: Expert

at the beginning and end of Fall 19 semester, and again at the beginning and end of Fall 21 semester.

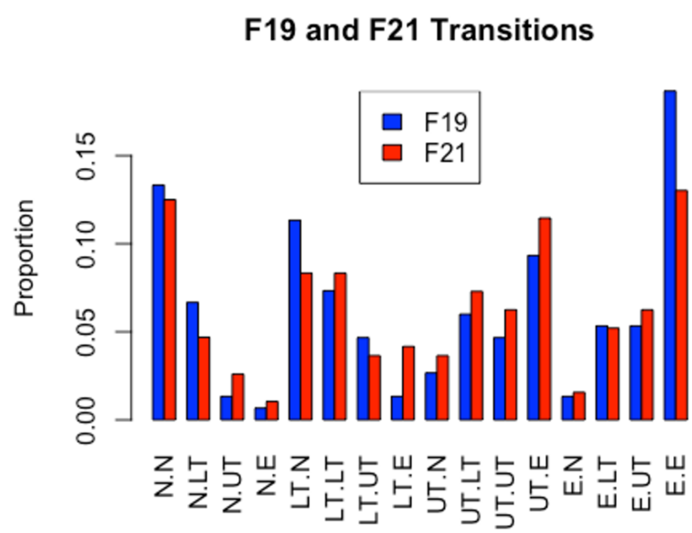


We pose the question, "*Does the change in students' VAMS profiles differ for students in F19 and F21?* " . In Fall 19, we notice a decrease in the E profile, with a corresponding increase of all the other profiles. The largest increase is for the profile UT. Similarly, for Fall 21, there is a decrease of the profile E, but the profile UT seems to grab most of the flow out of E, while the profile LT decreases. We run $\chi^2$ tests to compare distributions:

| Distributions | p-value |
|---|---|
| F19-Start vs F19-End | 0.389 |
| F21-Start vs F21-End | 0.134 |
| F19-Start vs F21-Start | 0.603 |
| F19-End vs F21-End | 0.901 |

We are also interested in recording individual transitions during the semesters, and, in the following table, we count the number of students who made the corresponding transitions. Since there are 4 initial profiles and 4 terminal profiles, there are 16 possible transitions. These transitions can be visualized with the following barplot. For clarity, N.UT means the student started the corresponding semester with an N profile and ended the same semester with an LT profile.




We notice that in F21 students were more likely than in F19 to stay in the initial profile [N.N, LT.LT, UT.UT, E.E] with the exception of UT.UT. There was a higher transition out of the profile N in F21 than in F19. A large percentage of those starting in LT tend to stay in LT or move to N, this is even more evident in F21 than in F19.

We run a $\chi^2$ test to check whether the distribution of transitions in F19 and in F21 are different. The p-value is quite high.

X-squared = 7.9442, df = 15, p-value = 0.926

Another way to see the individual transitions during the semesters is to look at the percentage of students who stayed in the same category, the percentage of students who went "up" one or more category (e.g. N to LT), and the percentage of students who went "down" one or more category (e.g. E to LT). These percentages are summarized in the following table:

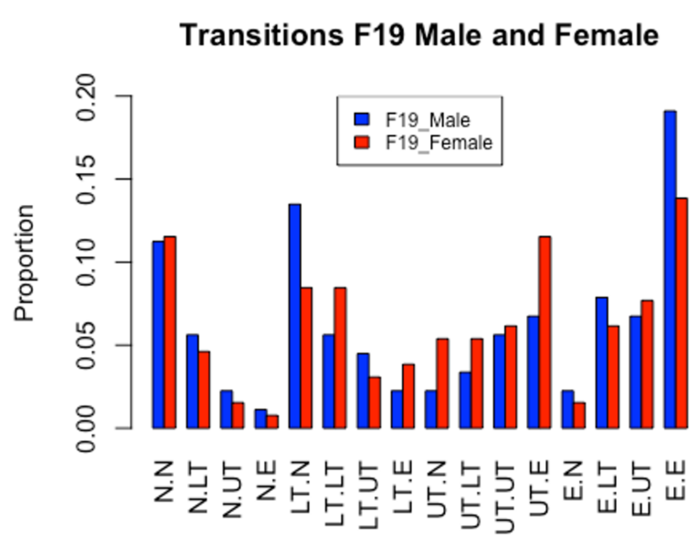
| | Same | Down | Up |
|---|---|---|---|
| F19 | 40.00% | 34.00% | 26.00% |
| F21 | 43.23% | 30.73% | 26.04% |

Here, we can see that the percentage of students who went up is surprisingly close, but in Fall of 19 we had a higher proportion of students move down than we did in Fall of 21. There is no statistical difference between the distributions in the table above, as we can see from a $\chi^2$ test:

X-squared = 0.29057, df = 2, p-value = 0.8648

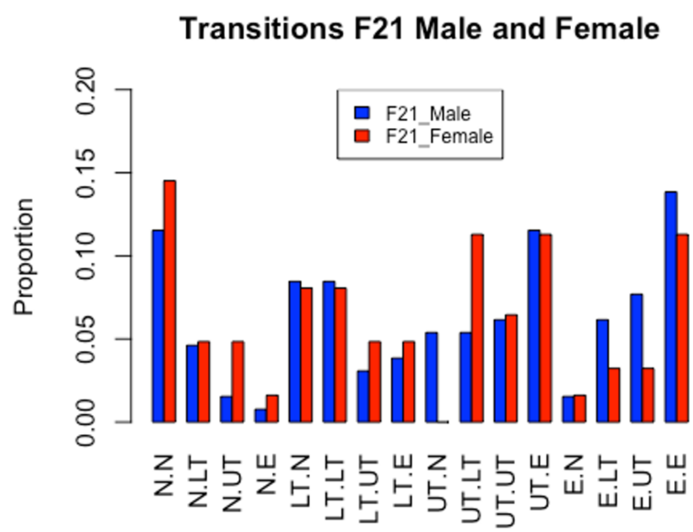
**Separation by Gender**

We are also interested in the analysis of the data based on gender. We pose the question: "*Does the change in students' VAMS profiles differ for male and female students in F19 and/or F21?*". As reported above the numbers are for F19: Males 96, Females 54; for F21: 123 males, 69 females. For each semester, we look at transitions during the same semester for males and females. We first include two bar plots, one for F19, and one for F21, comparing males and females.



In F19, females were more likely than males to move to the right [improve their profile]: N-LT, LT-UT, UT-E. Among males, we notice the high LT-N transition, and the relative high percentage of transition E-LT. A large percentage of females remained in N. Due to low values in some entries (e.g. N.E), instead of a $\chi^2$ test, we run a Fisher's Exact test [15][16] with

p-value = 0.4668




In F21, for females, we notice the transition N-E [0% in F19], no UT-N transition, and limited transitions E-N, E-LT, E-UT. There is also a spike in UT-LT, which shows up in F19 as well. For males, we see a more likely shift from E to UT and LT, and a larger UT-E. The Fisher's exact test yields

p-value = 0.7466

As before, we can also see the individual transitions during the semesters by looking at percentage of students of each gender who stayed in the same category, the percentage of students who went "up" one or more category (e.g. N to LT), and the percentage of students who went "down" one or more category (e.g. E to LT). These percentages are summarized in the following table:

| | Same | Down | Up |
|---|---|---|---|
| F19 Male | 37.50% | 38.54% | 23.96% |
| F19 Female | 44.44% | 25.93% | 29.63% |
| F21 Male | 41.00% | 31.00% | 28.00% |
| F21 Female | 43.48% | 27.54% | 28.99% |

Repeated $\chi^2$ tests show no significant difference among the above distributions.

| Distributions | p-value |
|---|---|
| F19 M vs F19 F | 0.1609 |
| F19 M vs F21 M | 0.5253 |
| F19 F vs F21 F | 0.9676 |
| F21 M vs F21 F | 0.8631 |

There does not seem to be significant changes here, but we can see that in Fall 21, fewer male students moved down, and more moved up than in Fall 19. The differences for female students in these two semesters are much more subtle but seem to be in the opposite direction from the male students.

We observe that these VAMS results include students from an additional section/instructor in Fall of 21. That is, there were two instructors for the surveyed students in Fall of 19, and while these professors' students were also surveyed in Fall of 21, so were students from another professor's section.

**Limitations**

In interpreting results, we note that Fall 2021 was the first semester of in-person instruction after the Covid pandemic. It was not uncommon for students to miss a week of class or to have to try participating remotely, due to contracting or being in contact with the virus. It was also the first semester of full implementation of the IODE curriculum. Based on results in other Applied Math courses that semester, we would expect to see a loss of procedural fluency in the Fall 2021 semester. Unfortunately, we cannot determine how much our results were influenced by factors related to the pandemic. Additionally, our results are limited by the small number of assessment questions used for comparison across the Fall 2019 and 2021 semesters.

**Conclusion**

This study reports on the implementation of an inquiry-oriented curriculum in Ordinary Differential Equations course for engineering students. The curriculum is designed to introduce mathematical content with real world problems, teach students to *construct* differential equations that model physical situations, connect analytic, numerical, and graphical representations and solution methods, and facilitate discussion and argumentation in problem solving. Data measuring student beliefs and performance from two semesters, with and without the new curriculum, was collected and analyzed.

With the Bonferroni correction, we saw that the improvement in performance on conceptual questions between Fall of 2019 and Fall of 2021 was not quite enough to be statistically significant. If we had run fewer tests, this result would have been statistically significant. At the same time, we do not have evidence that the IODE curriculum negatively impacts students' procedural fluency. This is particularly encouraging, since the IODE curriculum was in its first semester of implementation and was implemented during the pandemic. We hope and expect that through further iterations of the curriculum implementation, student performance will improve and extend well beyond where it was in Fall 2019. In the future, we plan on repeating more test questions from Fall of 2019 so that we can see what the change is once the newly designed class has been refined, and the impact of the pandemic on students' performance is reduced.

Student beliefs about mathematics were measured at the start and end of the two semesters using the Views About Mathematics Survey. We saw a trend away from the expert view and towards the naïve view from the beginning of the semester. This is consistent with previous findings which suggested this trend may be explained partially by historically strong mathematics students struggling to maintain confidence and persistence in their first college mathematics course [13][14]. This trend did not differ significantly between the two semesters. We also did not find a significant difference in the change of views between male and female students.

Data will be collected in future semesters to compare the performance of students in the IODE curriculum after it has been further modified and refined. We also hope to analyze responses to specific VAMS items in future semesters and compare these responses to our existing data.

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Worksheet sequence for solving a first order linear equation with an integrating factor.

Worksheet 1: Pre-class Worksheet due September 6, page 1

| **Name:** |
|---|
| |

**APMA 2130 - Pre-Class Worksheet: Mixing Problem and Using the Product Rule**
**Date:**

1. Consider the situation:

   A large tank initially contains 50 grams of salt in 10 liters of water. Seawater with a concentration of 40 grams of salt per liter is dumped into the tank at a rate of 3 liters per minute. The tank is *well mixed* and is drained at a rate of 5 liters per minute

   Let $S(t)$ represent the number of grams of salt in the tank at time $t$.

   (a) How long will it take for there to be no liquid left in the tank?

   **Solution:** There are 10 liters of water in the tank, and it is being drained two liters per minute faster than it is being filled. Thus, it is draining at a rate of 2 liters per minute, and it will be empty in five minutes.

   (b) Give an IVP for $S(t)$.

   **Solution:** We know that salt is coming into the tank at a rate of $40\frac{\text{g}}{\text{L}} \cdot 3\frac{\text{L}}{\text{min}} = 120\frac{\text{g}}{\text{min}}$
   The concentration of salt in the tank at time $t$ is $\frac{S(t)\text{ g}}{(10-2t)\text{ L}}$.
   Since this is a well mixed solution, the concentration of salt in the tank at time $t$ is the same as the concentration of salt leaving the tank at time $t$. Thus, salt is leaving the tank at a rate of $\left(\frac{5 \cdot S(t)}{10-2t}\right)\frac{\text{g}}{\text{min}}$
   Putting everything together, we have the IVP:

   $$\boxed{\frac{dS}{dt} = 120 - \frac{5 \cdot S(t)}{10-2t}, \quad S(0) = 50}$$

   (c) For what values of $t$ will the solution for the IVP actually model the amount of salt in the tank at time $t$?
   Explain your answer.

   **Solution:** It will only work for $0 \leq t < 5$ because after 5 minutes, there is no liquid (or salt) left in the tank, and we don't know what happens with the amount of salt in the tank before $t = 0$ (although it's likely just a constant 50 grams).

   (d) Can you solve the IVP using methods we've covered in class so far? If you can, do it here. If not, explain why.

   **Solution:** You should not be able to solve it. It's not possible to get a function of $S$ multiplied by $\frac{dS}{dt}$ with a function of only $t$ on the left hand side of the equation.

2. Suppose that $y$ is a function of $t$. Differentiate the following expressions with respect to $t$:

(a) $y \sin t$

**Solution:** $\frac{d}{dt}[y \sin t] = \sin t \cdot y'(t) + \cos t \cdot y(t)$

(b) $t^2 y$

**Solution:** $\frac{d}{dt}[t^2 y] = t^2 y'(t) + 2t \cdot y(t)$

(c) $t \cdot y$

**Solution:** $\frac{d}{dt}[t \cdot y] = t \cdot y'(t) + y(t)$

(d) $\frac{y}{t}$

**Solution:** $\frac{d}{dt}\left[\frac{y}{t}\right] = \frac{y'(t)}{t} - t^{-2}y(t)$

3. Consider the equation $t^2 y' + 2ty = 1$.

(a) Using your work from question 2, write the left hand side as a derivative of one expression.

**Solution:**

$$t^2 y' + 2ty = \frac{d}{dt}\left[t^2 y\right] \implies \boxed{\frac{d}{dt}\left[t^2 y\right] = 1}$$

(b) Now that the left hand side is a derivative, integrating both sides is the next logical step. So do that here.

**Solution:**

$$\frac{d}{dt}\left[t^2 y\right] = 1 \implies \int \frac{d}{dt}\left[t^2 y\right]\, dt = \int 1\, dt \implies \boxed{t^2 y = t + C}$$

(c) Solve for $y$ to find the **general solution** to this differential equation.

**Solution:**

$$t^2 y = t + C \implies \boxed{y = \frac{t + C}{t^2}}$$

| **Names of Present Group Members:** | **Group #:** |
| --- | --- |
| | |

**APMA 2130 - Worksheet: Linear Equations**
**Date:**

1. In your pre-class worksheet, you solved the first order linear equation: $t^2y' + 2ty = 1$

   This equation was especially easy to solve because we had that $\frac{d}{dt}\left[t^2y(t)\right] = t^2y'(t) + 2ty(t)$, so we could write the left hand side as a derivative using the product rule.

   Now, consider the equation $t^2y' + 3ty = t^2$

   (a) By which function could we multiply both sides of this equation so that the left hand side is a derivative of the product of $y$ and another function?

   **Hint:** In the last method we covered, we though about writing the one side as a derivative using the chain rule. Now we want to think writing one side as a derivative using the product rule.

   > **Solution:** Hopefully, it shouldn't be too hard to guess that multiplying the equation by the function $t$ will allow the left hand side to be a derivative of $y(t)t^3$

   (b) The function you found in (a) is called an **integrating factor** and is typically denoted by $\mu(t)$ (or $\mu$ of whatever the independent variable is for the given ODE).

   Multiply through by the function from part (a) - the integrating factor $\mu(t)$ - and rewrite the new left hand side as a derivative of the product of $y$ and another function.

   Check your work by differentiating.

   > **Solution:** After multiplying by $\mu(t) = t$, we have the updated, but equivalent equation:
   > $$t^3y' + 3t^2y = t^3$$
   > The left hand side here is equivalent to: $\frac{d}{dt}\left[t^3y(t)\right] = t^3y'(t) + 3t^2y(t) \quad \checkmark.$

   (c) Find the general solution to the differential equation.

   > **Solution:** After multiplying through by $t$, we have that the ODE $t^2y' + 3ty = t^2$ can be solved as follows:
   > $$\begin{aligned} t^3y' + 3t^2y &= t^3 \\ \frac{d}{dt}\left[t^3y(t)\right] &= t^3 \\ \int \frac{d}{dt}\left[t^3y(t)\right]dt &= \int t^3\,dt \\ t^3y(t) &= \tfrac{1}{4}t^4 + C \\ y(t) &= \tfrac{1}{4}t + Ct^{-3} \end{aligned}$$
   > Thus, the general solution to the ODE is: $\boxed{y(t) = \frac{1}{4}t + Ct^{-3}}$

In this case, the integrating factor was (hopefully) particularly easy to find by "inspection" (fancy word for guessing). However, this is not usually the case. On the next page, you'll be introduced to a process of solving a differential equation for $\mu$ in order to find it.

**Definitions:**

- Recall that the **order** of an ordinary differential equation is the highest derivative of the solution/dependent variable that occurs in the differential equation.
- We call a first order equation with solution variable $y$ and independent variable $t$ **linear** if we can write it in one of the following forms:

$$P(t)y' + R(t)y = G(t) \quad \text{or} \quad y' + p(t)y = g(t)$$

- A first order linear equation is considered to be in **standard form** when it is written in the format:

$$y' + p(t)y = g(t)$$

2. Consider the ODE: $x^3y' + 3xy = x^2$

(a) Put the the original equation in standard form and then multiply through through by the currently unknown function $\mu(x)$.

**Solution:** $\mu(x)y' + \mu(x)3x^{-2}y = \mu(x)x^{-1}$

(b) What differential equation should $\mu(x)$ satisfy so that the left hand side of the ODE from part (a) can be written as a derivative using the product rule?

**Solution:** Since the left hand side is $\mu(x)y' + \mu(x)3x^{-2}y$, we need: $\boxed{\mu'(x) = \mu(x)3x^{-2}}$

This will give us: $\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)y'(x) + \mu'(x)y(x) = \mu(x)y' + \mu(x)3x^{-2}y.$

(c) Solve the differential equation from part (b) to find <u>**one**</u> function that works for the integrating factor. Simplify your expression so it will be easier to work with.

**Solution:**

$$\begin{aligned} \frac{d\mu}{dx} &= 3x^{-2}\mu \\ \frac{1}{\mu}\frac{d\mu}{dx} = \frac{d[\ln(\mu)]}{d\mu}\frac{d\mu}{dx} &= 3x^{-2} \\ \frac{d}{dx}\left[\ln(\mu)\right] &= 3x^{-2} \\ \ln(\mu) &= -3x^{-1} \implies \mu = e^{-3/x} \\ \mu &= e^{-3/x} \end{aligned}$$

(d) Use the integrating factor to find the general solution for the ODE. Your answer will involve an integral that you cannot solve by hand, so it should include an indefinite integral.

**Solution:**

$$\begin{aligned} e^{-3/x}y' + 3x^{-2}e^{-3/x}y' &= e^{-3/x}x^{-1} \\ \frac{d}{dx}\left[e^{-3/x}y\right] &= e^{-3/x}x^{-1} \\ e^{-3/x}y &= \int e^{-3/x}x^{-1}\,dx \implies \boxed{y = e^{3/x}\int \frac{e^{-3/x}}{x}\,dx} \end{aligned}$$

**Name:**

**APMA 2130 - Pre-Class Worksheet: Linear Homogeneous and Nonhomogeneous Equations**

**Date:**

1. In the previous pre-class worksheet, you should have found the following ODE for a mixing problem:

$$\frac{dS}{dt} = 200 - 3 \cdot \frac{S}{4+2t}$$

We talked about this ODE in class, but we didn't get a chance to solve it. This question will take you through the steps needed for solving the ODE.

(a) What differential equation should the integrating factor, $\mu(t)$ satisfy for this equation?
**Hint:** To start, I'd advise rewriting the equation so that it is in standard form.

**Solution:** Start with: $\frac{dS}{dt} + 3 \cdot \frac{S}{4+2t} = 200$. Multiplying through by $\mu(t)$, we get:

$$\mu(t)\frac{dS}{dt} + 3\mu(t) \cdot \frac{S}{4+2t} = 200\mu(t) \implies \boxed{\frac{d\mu}{dt} = \frac{3\mu(t)}{4+2t}}$$

(b) Solve the differential equation from part (a) to find **one** function that works for the integrating factor. Simplify your expression so it will be easier to work with.

**Solution:**

$$\frac{d\mu}{dt} = \frac{3\mu(t)}{4+2t}$$

$$\frac{1}{\mu}\frac{d\mu}{dt} = \frac{d[\ln(\mu)]}{d\mu}\frac{d\mu}{dt} = \frac{d}{dt}\left[\ln(\mu)\right] = \frac{3}{4+2t}$$

$$\int \left(\frac{d}{dt}\left[\ln(\mu)\right]\right) dt = \int \frac{3}{4+2t}\, dt$$

$$\ln(\mu) = \tfrac{3}{2}\ln(4+2t) = \ln[(4+2t)^{3/2}] \implies \mu(t) = (4+2t)^{3/2}$$

(c) Use the integrating factor to find the general solution for the ODE.

**Solution:**

$$(4+2t)^{3/2}\frac{dS}{dt} + 3(4+2t)^{3/2} \cdot \frac{S}{4+2t} = 200(4+2t)^{3/2}$$

$$(4+2t)^{3/2}\frac{dS}{dt} + 3(4+2t)^{1/2} \cdot S = 200(4+2t)^{3/2}$$

Check product rule: $\frac{d}{dt}\left[(4+2t)^{3/2}\right] = \frac{3}{2}(4+2t)^{1/2} \cdot 2 = 3(4+2t)^{1/2}$ ✓

$$\frac{d}{dt}\left[S \cdot (4+2t)^{3/2}\right] = 200(4+2t)^{3/2}$$

$$S \cdot (4+2t)^{3/2} = \int 200(4+2t)^{3/2}\, dt = 200(4+2t)^{5/2}\left(\tfrac{2}{5}\right)\left(\tfrac{1}{2}\right) + C = 40(4+2t)^{5/2} + C$$

Thus, we have:

$$S(t) = \frac{40(4+2t)^{5/2} + C}{(4+2t)^{3/2}} \implies \boxed{S(t) = 40(4+2t) + C(4+2t)^{-3/2}}$$

Other Conceptual Test Questions from Fall 2019:

Test 1:

2. [4 pts] Suppose that the volume of water $V(t)$ in a spherical raindrop **decreases** (evaporates) at a rate **proportional** to its surface area $S(t)$.

   Find the relationship between $V(t)$ and $S(t)$, and use it to write an equation for $r'(t)$, the rate of change for the radius.

   **Hint:** The volume of a sphere is $\frac{4}{3}\pi r^3$, and the surface area of a sphere is $4\pi r^2$.

   **Second Hint:** Don't forget to use the chain rule.

Test 2:

8. Consider the IVP: $y' = y(y-3)^2(y+5), \quad y(2) = -1$
   (a) [5 pts] Make a phase line for all the solutions of the equation $y' = y(y-3)^2(y+5)$. **Be sure to box the phase line to make it clear that you know which one the phase line is.**

   (b) [2 pts] Suppose $f(t)$ is a solution of the IVP: $y' = y(y-3)^2(y+5),\ y(2) = -1$. Find $\lim_{t\to\infty} f(t)$, and justify your answer.

Test 3:

7. [5 pts] Is $s = 2$ in the domain of $\mathcal{L}\left\{e^{2t}\frac{1}{\sqrt{t+3}}\right\} = F(s)$? If so, give the value of $F(2)$. If not, explain why.

9. [4 pts] Is $f(x) = \cos(x^2)$ a solution of $xy'' - y' + 4x^3y = 0$?
   Justify your answer by showing all necessary work.