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An Alternative Methodical Approach and Its Effectiveness to Learn Change of Basis Matrices in an Engineering Linear Algebra Class

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Abstract

Over the years, students have relied on textbook approaches to learn change of basis matrices. These methodologies often stem from complex abstract concepts, making them difficult for students to comprehend, especially for engineering students who do not have sufficient training for proofs. As a result, lots of students frequently resort to rote memorization of formulas without truly understanding the concepts behind them. Due to the intricacy of these formulas, it's not uncommon for students to make mistakes, leading to poorly solved problems. In response to this issue, this paper introduces an alternative approach that diverges from traditional textbook methods and investigates students' perceptions on the effectiveness of this alternative approach. This alternative approach builds on foundational linear algebra skills, making it considerably more accessible and easier to understand for students. Furthermore, it eliminates the need for students to memorize complex formulas, promoting sustainable, long-term learning instead.

Keywords: linear algebra, linear transformation matrix, change of basis, change of basis matrix, alternative approach

Introduction

Linear algebra is a fundamental area of study that has applications in a wide range of disciplines. It provides the principles and techniques necessary for analyzing vector spaces, linear transformations, and systems of linear equations. This knowledge serves as a foundation for advanced studies in fields such as computer science, physics, engineering, data science, economics, and more. Additionally, it equips individuals with essential tools for data analysis, including tasks like dimensionality reduction, data compression, and data visualization. These techniques are particularly important in areas like machine learning, data mining, and signal processing, where concepts from linear algebra, such as singular value decomposition (SVD) and principal component analysis (PCA), form the basis for advanced methods.

One of the most critical topics in a linear algebra curriculum is the concept of change of basis. This concept allows for the understanding and analysis of vectors and matrices from different perspectives or coordinate systems. By exploring various coordinate systems, we can gain insights into the underlying structure of problems, especially in physics, where different coordinate systems like Cartesian, polar, or spherical coordinates offer different viewpoints. Change of basis also enables the analysis of transformations between different coordinate systems, both linear and non-linear. By utilizing a new basis, we gain a distinct perspective that helps us comprehend the characteristics and properties of the transformation, which is particularly relevant in computer graphics, robotics, and control theory. To learn the concept of change of basis, change of basis matrix (CBM) is the foundation since it defines the specific change from one coordinate system to another coordinate system.

There are literatures exploring different approaches, practices, and applications for linear algebra concepts such as "span" [1], [2], [3], [4], [5] , "linear independence" [1], [2], [4], [6], [7], [8],

[9], and “eigenvalues/eigenvectors” [10], [11], [12], [13], [14], [15]. There are also researches on pedagogical innovations of teaching linear algebra with or without programming technology incorporated into the course to reinforce students’ understanding [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28]. However, there's little research focusing on the learning methods for CBM. Over the years, this topic is typically taught using abstract theoretical methods as described in textbooks [29], [30], [31], [32], [33], [34], [35], [36]. While the textbook approaches mentioned are commendable, they often assume a high level of understanding of abstract concepts and rely on strong logical skills. As a result, these approaches may be more suitable for mathematics majors rather than engineering students.

APMA 3080-Linear Algebra is a course specifically targeted for students in the School of Engineering & Applied Science at the University of Virginia. The dominant majority students enrolled are engineering majors, with occasional a few participants from other majors as well. It is important to consider that engineering students may not have received as extensive training in logical reasoning and mathematical principles as their counterparts in mathematics programs. Consequently, the traditional textbook approaches used in the course can present challenges for instructors trying to effectively convey these complex topics. Over the years of teaching Linear Algebra with these methods, it has become evident that many students struggle to grasp these concepts fully, often resorting to rote memorization and occasionally becoming confused, which ultimately hinders their overall understanding.

As an experienced instructor and the course coordinator for APMA 3080-Linear Algebra, I have been determined to find alternative teaching strategies to address the challenges surrounding the topic of change of basis matrix (CBM). This paper aims to present an alternative methodical approach that can enhance students' understanding of CBM. Additionally, it explores students' perceptions of the effectiveness of this approach, offering potential benefits to other educators facing similar difficulties in teaching this concept.

Table 1: Abbreviation

Concepts	Abbreviation
Linear Transformation	LT
Linear Transformation Matrix	LTM
Linear transformation from R^n to R^m	LT: $R^n \rightarrow R^m$
Change of Basis Matrix	CBM

The paper is divided into four sections. The first section covers necessary preliminary linear algebra knowledge that is standard for the class but required to master the alternative method. The second section reviews the conventional textbook method that has been employed in the past. The third section provides a comprehensive explanation of the alternative approach being proposed. Lastly, the fourth section focuses on exploring the perceptions of students regarding this alternative method. To enhance clarity and convenience, an accompanying table (Table 1) has been included, which outlines useful shortcuts for reference.

Preliminary

The alternative approach starts from how to find LTM: $R^n \rightarrow R^m$, which is relatively straightforward and well established in APMA 3080- Linear Algebra at the School of Engineering at our institution. In this section, I present the way of finding LTM: $R^n \rightarrow R^m$ and some basic definitions as prerequisites to master the alternative approach. Such prerequisites are usually standard topics in a linear algebra class, so there is no extra work if adapting the alternative approach to find CBM.

Linear transformation matrices for linear transformations from R^n to R^m

Definition 1 [37]*:* A function $T: R^n \to R^m$ is a *linear transformation* if for all vectors $\vec{u}, \vec{v} \in R^n$ and all $s \in R$ we have: (1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, and (2) $T(s\vec{u}) = sT(\vec{u})$.

Theorem 1 [37]*:* Let $T: R^n \to R^m$. Then it is a linear transformation if and only if $T(\vec{x}) = A\vec{x}$ for some $m \times n$ matrix A.

Definition 2 [37]*:* The matrix A in Theorem 1 is called the *linear transformation matrix* for a linear transformation $T: R^n \to R^m$.

Common approach to find LTM: $R^n \to R^m$: $A = [\, T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \ldots \;\; T(\vec{e}_n) \,]$, where $\vec{e}_i \in R^n$ with the *i-th* component to be 1 and all else to be 0, namely,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix},\ \cdots,\ \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

The set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is also called a *standard basis* of R^n.

Example 1: Given a linear transformation $T: R^3 \to R^2$ such that $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 - x_1 \end{bmatrix}$, find its associated linear transformation matrix A such that $T(\vec{x}) = A\,\vec{x}$.

Solution: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then $T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T(\vec{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0+0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Consequently, $A = [\, T(\vec{e}_1) \;\; T(\vec{e}_1) \;\; T(\vec{e}_3) \,] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. □

This method for finding LTM: $R^n \to R^m$, which is prevalent in many textbooks [29], [30], [31], [32], [33], [37], has a well-established rationale. Since students typically grasp this method effectively, we will not delve into the underlying logic in this paper. However, it's important to note that this method to obtain LTM: $R^n \to R^m$ is fundamental to the alternative approach discussed in subsequent sections of finding CBM.

Coordinate vector of $\vec{x}$ with respect to a basis B

Here we review the definition from the widely used textbook "Linear Algebra and its Application" by David C. Lay et al [29]. The definitions from other textbooks are very similar except with slight difference for names and symbols.

Definition 3 [29]***:*** Assume $B = \{\vec{u}_1, \ldots, \vec{u}_n\}$ is a basis of the vector space V, *the B-coordinates of* $\vec{x}$ (*the coordinates of* $\vec{x}$ *with respect to the basis B*) are the coefficients in the linear combination of $\vec{x}$: $\vec{x} = c_1\vec{u}_1 + \ldots + c_n\vec{u}_n$. Let $[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, then $[\vec{x}]_B$ is the *coordinate vector of* $\vec{x}$ *(with respect to B)*, or *the B-coordinate vector of* $\vec{x}$.

Each basis of a given vector space can viewed as a coordinate system. Take R^3 for example, the most familiar coordinate system in R^3 is xyz-coordinate system which corresponds to the standard basis $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of R^3. If a different basis $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ of R^3 is used, it will make a different coordinate system and any vector $\vec{x}$ in this different coordinate system will have different coordinates. If $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$, it can be interpreted as: $\vec{x}$ takes c_1 units in the direction of $\vec{b}_1$, $\vec{x}$ takes c_2 units in the direction of $\vec{b}_2$, and $\vec{x}$ takes c_3 units in the direction of $\vec{b}_3$. Then the coordinates of $\vec{x}$ with respect to the basis B are (c_1, c_2, c_3). Use such coordinates to compose a vector $[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, which is the coordinate vector of $\vec{x}$ with respect to *B (B-*coordinate vector*)*. Please refer to the book for more details about mathematical and geometrical interpretation.

Change of basis matrix

Let's simply review the definition of "change of basis matrix" in various textbooks [29], [30], [31], [32], [33], [35]. There may be slight difference for wordings, but the meanings are the same.

Definition 4 [37]*:* If $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is a basis of R^n, let $U = \left[\vec{b}_1 \ \vec{b}_2 \ \ldots \ \vec{b}_n\right]$, then U is called the *change of basis matrix* from the basis B to the standard basis S in R^n.

Consider a vector $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \in R^n$, then its B-coordinate vector of $\vec{x}$, $[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, and $U[\vec{x}]_B = \left[\vec{b}_1 \ \vec{b}_2 \ \ldots \ \vec{b}_n\right] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{x} = [\vec{x}]_S$. This matrix U takes B-coordinates of any vector $\vec{x}$ to the standard coordinates by multiplying it from the left, *so* change of basis matrix is also called *change-of-coordinates matrix***.** An analogous CBM can be carried out in R^n from S to B, or from one nonstandard basis B_1 to another nonstandard basis B_2.

Textbook Method to Find CBMs

In many textbooks, the process of determining the CBMs from one basis to another relies on a mix of linear algebra concepts and robust logical reasonings. We will examine the approach demonstrated in Jeffrey Holt's book [37] for a clearer picture.

Let $\mathcal{B}_1 = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}_2 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for $\mathbf{R}^n$. If $U = [\mathbf{u}_1 \ldots \mathbf{u}_n]$ and $V = [\mathbf{v}_1 \ldots \mathbf{v}_n]$, then

(a) $[\mathbf{x}]_{\mathcal{B}_2} = V^{-1}U[\mathbf{x}]_{\mathcal{B}_1}$

(b) $[\mathbf{x}]_{\mathcal{B}_1} = U^{-1}V[\mathbf{x}]_{\mathcal{B}_2}$

Figure 1: Textbook method of how to find CBM from one basis to another basis.

This commonly used method identifies $V^{-1}U$ as the CBM from B_1 to B_2, and $U^{-1}V$ as the CBM from B_2 to B_1, as stated in Figure 1. Although this approach has been employed for years and is theoretically accurate, it has been observed that students often make errors, especially among engineering students even if they initially grasp the concept of why CBMs should be calculated in this manner. In addition, the level of understanding among students greatly relies on how teachers explain the method. Experienced teachers are better at making things clear, which helps students grasp the concepts more easily.

During my experience teaching this topic, in the earlier years, I would present the proof to demonstrate why the CBM could be found in this way. However, I found that the proof required strong logical skills, and some students found it challenging to follow. I had to repeat the proof multiple times for students to fully grasp it. Even for those students who initially understood the proof, their comprehension often became blurry after a few days. As a result, lots of students turned to memorize the method without any understanding. However, the difficulties persisted even only with memorizations. Common issues among students include confusion regarding the order of matrix multiplication, the direction of the CBM, the placement of the inverse, and the identification of matrices U and V when the base order and symbols were altered.

Alternative Method to Find CBMs

The core idea of the alternative approach is viewing a CBM as a LTM and use the common approach to find the LTM to get the CBM. By Theorem 1, $T: R^n \to R^m$ is a LT if and only if $T(\vec{x}) = A\vec{x}$ for some matrix A. By Figure 1, $[\vec{x}]_{B_2} = V^{-1}U\,[\vec{x}]_{B_1}$, so it can be viewed as a linear transformation $T: R^n \to R^n$ such that $T\big([\vec{x}]_{B_1}\big) = V^{-1}U\,[\vec{x}]_{B_1}$ with the associated LTM to be $V^{-1}U$. This transformation transforms the coordinates of a vector in the B_1-coordinate system to the coordinates in the B_2-coordinate system.

In the case that we need to find CBM from B_1 to B_2, we treat it as finding LTM for the linear transformation $T\big([\vec{x}]_{B_1}\big) = [\vec{x}]_{B_2}$. and we mention B_1 as the input basis for the input $[\vec{x}]_{B_1}$ and B_2 as the output basis for the output $[\vec{x}]_{B_2}$. Then CBM =[$T(\vec{e}_1)$ $T(\vec{e}_1)$ ... $T(\vec{e}_n)$], where $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ are treated as coordinate vectors with respect to B_1. In the case we need to find CBM from B_2 to B_1, we treat it as finding LTM for the linear transformation $T\big([\vec{x}]_{B_2}\big) = [\vec{x}]_{B_1}$. and we mention B_2 as the input basis for the input $[\vec{x}]_{B_2}$ and B_1 as the output basis for the output $[\vec{x}]_{B_1}$. Then CBM =[$T(\vec{e}_1)$ $T(\vec{e}_1)$ ... $T(\vec{e}_n)$], where $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ are treated as coordinate vectors with respect to B_2.

For example, to find the CBM from B_1 to B_2, we need to find the LTM for $T: R^n \to R^n$ such that $T\big([\vec{x}]_{B_1}\big) = [\vec{x}]_{B_2}$.

Since $[\vec{x}]_{B_1}$ is the input, we set $[\vec{x}]_{B_1} = \vec{e}_i$ $(i = 1, \dots, n)$ and we need to find $T(\vec{e}_i)$. $T(\vec{e}_i) = T\big([\vec{x}]_{B_1}\big) = [\vec{x}]_{B_2}$, namely, we need to find $[\vec{x}]_{B_2}$.

$[\vec{x}]_{B_1} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, by the definition of coordinate vector (Definition 3), $\vec{x} = 1\,\vec{u}_1 + 0\,\vec{u}_2 + \cdots + 0\,\vec{u}_n = \vec{u}_1$. Hence, $\mathrm{T}(\vec{\mathrm{e}}_1) = [\vec{u}_1]_{B_2}$. Following similar process will lead to $\mathrm{T}(\vec{e}_i) = [\vec{u}_i]_{B_2}$.

Practical steps for the alternative approach

If $B_1 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $B_2 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are two different bases of R^n, then we can find the change of basis matrix from B_2 to B_1 (or B_1 to B_2) in the following way.

Step1: Recognize the direction of CBM. E.g., if the problem asks to find the CBM from B_2 to B_1 ($B_2 \to B_1$), then the associated LT is $T\left([\vec{x}]_{B_2}\right) = A\,[\vec{x}]_{B_1}$ and we need to find A. In other words, we need to find the LTM for $T: R^n \to R^n$ such that $T\left([\vec{x}]_{B_2}\right) = [\vec{x}]_{B_1}$. Note that here $[\vec{x}]_{B_2} = [\vec{x}]_{input\ basis}$, and $[\vec{x}]_{B_1} = [\vec{x}]_{output\ basis}$, and vice versa.

Step2: Use the common approach to write down LTM: $R^n \to R^n$: $A = [\,T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \ldots \;\; T(\vec{e}_n)\,]$.

Step 3: Find $T(\vec{e}_i)$. Set up the input to be the unit vector $\vec{e}_i = [\vec{x}]_{input\ basis}$, and we need to find the output $\mathrm{T}(\vec{\mathrm{e}}_{\mathrm{i}}) = [\vec{x}]_{output\ basis}$

i) Find $\vec{x}$ first, by using the definition of “coordinate vector” since the coordinate vector $[\vec{x}]_{input\ basis}$ is given with respect to the input basis.

ii) Then find the coordinate vector of $\vec{x}$ with respect to the “output” basis, by using the definition of “coordinate vector” again.

Finding $T(\vec{e}_i)$ only relies on the understanding for the definition of coordinate vectors and skills to solve linear systems.

Step 4: wrap up the answer, *CBM*=$[\,T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \ldots \;\; T(\vec{e}_n)\,]$.

Example 2: $B_1 = \{\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$, and $B_2 = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\}$ are two bases of R^2, find the change of basis matrix from B_2 to B_1.

Solution:
Step1. we need to find the CBM from B_2 to B_1, then the associated linear transformation is $T\left([\vec{x}]_{B_2}\right) = [\vec{x}]_{B_1}$, namely, we need to find A such that $A\,[\vec{x}]_{B_2} = [\vec{x}]_{B_1}$,

Step2. $\mathrm{A} = [\,\mathrm{T}(\vec{\mathrm{e}}_1) \;\; \mathrm{T}(\vec{\mathrm{e}}_2)\,]$, where $\{\vec{e}_1, \vec{e}_2\}$ are treated as B_2- coordinate vectors since the input of T is B_2- coordinate vectors.

Step3. find $\mathrm{T}(\vec{\mathrm{e}}_{\mathrm{i}})$. Set up $[\vec{x}]_{B_2} = \vec{\mathrm{e}}_{\mathrm{i}}$, then $\mathrm{T}(\vec{\mathrm{e}}_{\mathrm{i}}) = T\left([\vec{x}]_{B_2}\right) = [\vec{x}]_{B_1}$, namely, we need to find $[\vec{x}]_{B_1}$

1. Find $\vec{x}$ first. $[\vec{x}]_{B_2} = \begin{bmatrix}1\\0\end{bmatrix}$, which means the B_2-coordinate vector of $\vec{x}$ is $\begin{bmatrix}1\\0\end{bmatrix}$. By Def.3, $\vec{x} = 1\,\vec{v}_1 + 0\,\vec{v}_2 = \vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$
2. then find $[\vec{x}]_{B_1}$, by Def.3 again, we can get $[\vec{x}]_{B_1} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$ by solving the linear system $\begin{bmatrix}1\\1\end{bmatrix} = c_1 \begin{bmatrix}1\\2\end{bmatrix} + c_2 \begin{bmatrix}-1\\1\end{bmatrix}$, which is $\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2/3\\-1/3\end{bmatrix}$. Hence $\mathrm{T}(\vec{e}_1) = \begin{bmatrix}2/3\\-1/3\end{bmatrix}$.

Repeat the process, set $[\vec{x}]_{B_2} = \vec{e}_2$, we get $\mathrm{T}(\vec{e}_2) = \begin{bmatrix}\frac{5}{3}\\-\frac{1}{3}\end{bmatrix}$

Step 4. To sum up, the change of basis matrix from B_2 to B_1 is $\begin{bmatrix}\frac{2}{3} & \frac{5}{3}\\-\frac{1}{3} & -\frac{1}{3}\end{bmatrix}$. □

It is important to highlight that although the above solving process may seem lengthy, much of the writing can be shortened once students have fully mastered it. Furthermore, students typically only need to practice one to three examples to gain a complete understanding of the method, without memorizing any complicated formula.

This alternative approach can be easily extended to find the change of basis matrix in subspaces as well.

Example 3: $B_1 = \{\vec{u}_1 = \begin{bmatrix}1\\-5\\8\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}3\\-8\\3\end{bmatrix}\}$, and $B_2 = \{\vec{v}_1 = \begin{bmatrix}1\\-3\\2\end{bmatrix}, , \vec{v}_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix}, \}$ are two bases of a subspace of R^3, find the change of basis matrix from B_1 to B_2.

Solution:

*Step1***.** We need to find LTM for the linear transformation T: $R^2 \to R^2$ s.t $T\left([\vec{x}]_{B_1}\right) = [\vec{x}]_{B_2}$.

Step2. LTM $= [\ T(\vec{e}_1)\ \ T(\vec{e}_2)\ ]$, where $\{\vec{e}_1, \vec{e}_2\}$ are treated as B_1- coordinate vectors.

Step3. find $\mathrm{T}(\vec{e}_i)$. Set up $[\vec{x}]_{B_1} = \vec{e}_i$, then $\mathrm{T}(\vec{e}_i) = T\left([\vec{x}]_{B_1}\right) = [\vec{x}]_{B_2}$, namely, we need to find $[\vec{x}]_2$

1. Find $\vec{x}$ first. $[\vec{x}]_{B_1} = \begin{bmatrix}1\\0\end{bmatrix} \xRightarrow{Def.3} \vec{x} = 1\,\vec{u}_1 + 0\,\vec{u}_2 = \vec{u}_1 = \begin{bmatrix}1\\-5\\8\end{bmatrix}$
2. then find $[\vec{x}]_{B_2}$. by Def.3 again, we can get $[\vec{x}]_{B_2} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$ by solving the linear system $\begin{bmatrix}1\\-5\\8\end{bmatrix} = c_1 \begin{bmatrix}1\\-3\\2\end{bmatrix} + c_2 \begin{bmatrix}-1\\2\\1\end{bmatrix}$, which is $\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$. Hence $\mathrm{T}(\vec{e}_1) = \begin{bmatrix}3\\2\end{bmatrix}$.

Set up $[\vec{x}]_{B_1} = \vec{e}_2$, repeat the process, we get $\mathrm{T}(\vec{e}_2) = \begin{bmatrix}2\\-1\end{bmatrix}$.

Step 4. To sum up, the change of basis matrix from B_1 to B_2 is $\begin{bmatrix}3 & 2\\2 & -1\end{bmatrix}$. □

Effectiveness of the Alternative Method

Study design

A study was conducted to determine the effectiveness of the alternative approach presented in this research. Undergraduate students enrolled in APMA 3080 - Linear Algebra at the University of Virginia were recruited to participate in the study. Initially, the instructor taught the topic of change of basis using the textbook method, including the proof explaining why CBM could be calculated in that particular manner. Students then practiced solving problems to gain proficiency in the textbook method.

After three weeks of instruction, the instructor conducted a review of the textbook approach, during which students practiced solving a few problems using the textbook method again. Subsequently during the same class, the instructor introduced the alternative approach, covering the same topic, and students were given additional practice problems to solve using this approach.

After one week, students were asked (1) to solve some problems of CBM as a quiz and (2) to complete a survey using the Qualtrics platform. Students' solutions from (1) and the survey from (2) served as the main data sources for this study, supplanted by students' exam scores.

Regarding to the survey, it included seven Likert scale questions gauging students' perceptions of the alternative approach. Additionally, three multiple-choice questions were included to gather responses comparing the textbook method to the alternative method. Finally, a few open-ended questions provided students with the opportunity to provide written feedback on their perceptions and experiences.

The collected data was analyzed by statistical tools such as barplot, hypothesis tests, independent t-test, and proportion tests. Prior to applying these tests, we assessed the assumptions to ensure the most robust conclusions possible. The results are presented collectively for quantitative and categorized qualitative data, while qualitative data is reported individually. The management of collected data adheres strictly to the university's policy.

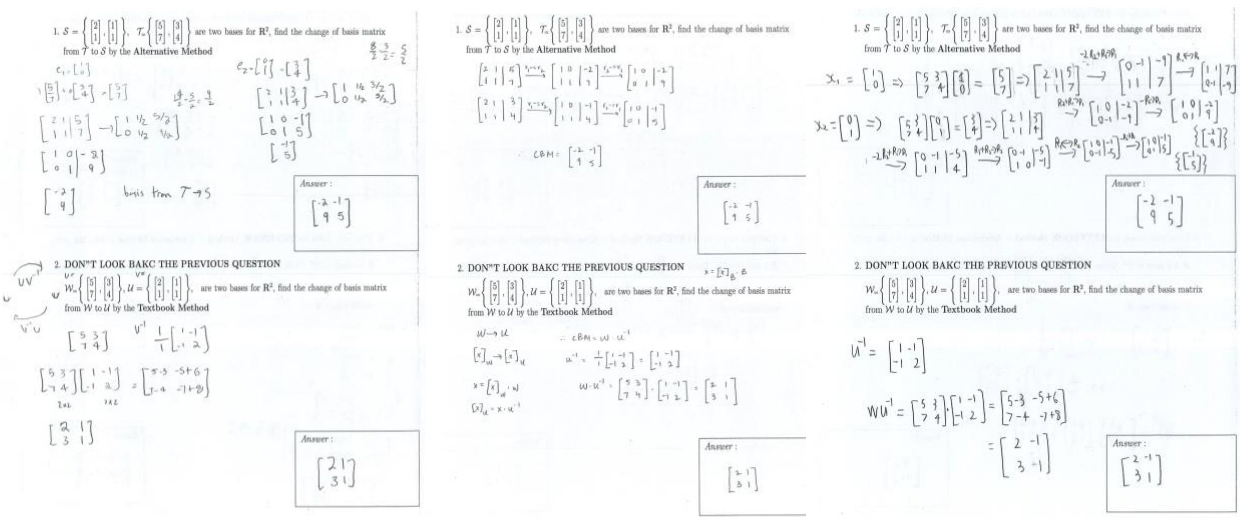
Students' exam scores comparison

We compared the supplementary data resource -exam scores specifically related to this topic, between students who were taught only using the textbook method and those who were taught with both methods.

In an exam from a previous semester where the alternative approach was not introduced, there was a test problem that tested the change of basis matrix. The average score for this problem was 6.1 (out of 10). However, after the introduction of the alternative approach, the same question was included in one exam using the same rubric to grade. The average score for this problem significantly improved to 8.2 (out of 10). An independent t-test was conducted, revealing that the students performed significantly better after being taught the alternative approach (8.2 vs 6.1, $p=0$) for this type of question.

Students' work comparison

As mentioned in the "Study design" section, after a week of learning both the textbook and alternative methods, students were given some problems to solve during an in-class quiz. The first quiz page had the same problem presented twice: once using the alternative method and once using the textbook method. Interestingly, many students solved the problem correctly using the alternative method but struggled with the textbook method. Below, you'll find solutions from a few students (Figure 2). Please enlarge them for better visibility.



Student A *Student B* *Student C*

Figure 2: Sample solutions from students for both the Textbook and Alternative Methods

Quantitative analysis and results of the students' perceptions

The survey's first part comprised seven questions, labeled Q1_1 through Q1_7, utilizing a 7-point Likert scale ranging from 1 (strongly disagree) to 7 (strongly agree). The bar graph (Figure 3) demonstrates that majority students agree that the alternative approach was easy to follow and less prone to errors compared to the textbook method. Furthermore, the alternative method was perceived as requiring less memorization and fewer logical deductions to fully comprehend. Students also expressed a higher level of comfort when applying the alternative method, coupled with a strong understanding of its concepts. Notably, for question Q1_7 regarding the overall effectiveness of the alternative method, only three students disagreed, eight students remained neutral, while majority students either somewhat agreed (18 students), agreed (35 students), or strongly agreed (21 students) (Figure 3).

It is notable that the average scores for the statements pertaining to the aspect of "memorization" (Q1_4) and the aspect of "logical skills requirement" (Q1_6) for the alternative are slightly below 5. This warrants further discussion. In a math class, strong skills are typically required, albeit at different levels. However, it is important to acknowledge that some students rely on memorization rather than understanding to learn math. It is possible that some students feel comfortable with and can apply the alternative approach, but their success is based on memorization rather than true comprehension. Although not explored in this analysis, it would be intriguing to ask students to rate the same statement for other topics, like "the textbook approach doesn't require strong logic skills." I hypothesize that the score would be lower in that case. While this aspect was not examined here, it would provide valuable insights for further analysis.

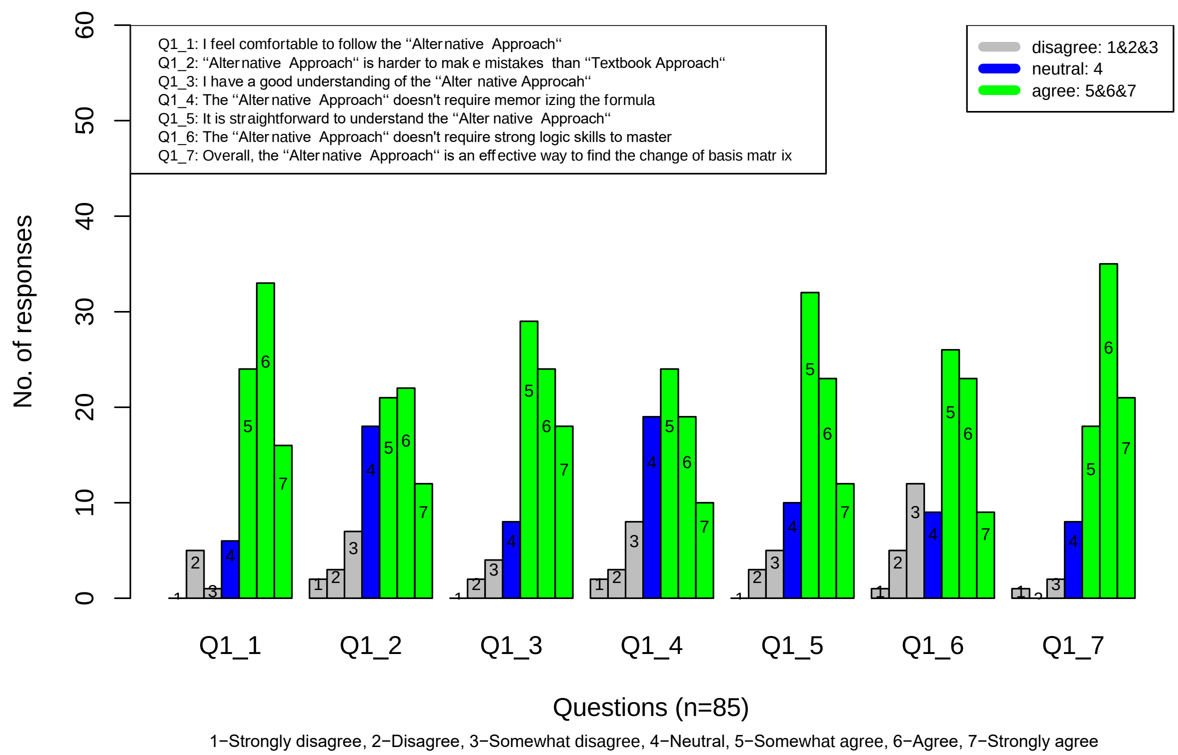


Figure 1: Students' perceptions on the Alternative Method

To draw statistically significant conclusions, the collected data was analyzed by hypothesis tests using various null hypotheses (H_0) and alternative hypotheses (H_a) based on the mean score values. For instance, the null hypothesis H_0: $\mu = 4$ assumes that students have a neutral stance on the statement, while the alternative hypothesis H_a: $\mu > 4$ suggests that students tend to somewhat agree with the statement. The results of these hypothesis tests are presented in Table 2.

Table 2: Hypothesis Tests on the mean score for statements about the Alternative Approach

	Mean	$H_0: \mu = 4$ vs $H_a: \mu > 4$ p-value	$H_0: \mu = 5$ vs $H_a: \mu > 5$ p-value	$H_0: \mu = 5.5$ vs $H_a: \mu > 5.5$ p-value
Q1_1 – “comfortable”	5.494	p<1.0*10^{-8} ***	.00013 **	NA
Q1_2 – “less mistake”	4.965	p<1.0*10^{-8} ***	NA	NA
Q1_3 – “understanding”	5.447	p<1.0*10^{-8} ***	.00029 **	NA
Q1_4 – “less memorization”	4.847	p<1.0*10^{-7} ***	NA	NA
Q1_5 – “straightforward”	5.212	p<1.0*10^{-8} ***	.049 *	NA
Q1_6 – “less logic skills”	4.871	p<1.0*10^{-8} ***	NA	NA
Q1_7 – “overall”	5.724	p<1.0*10^{-8} ***	p < 2.0*10^{-9} ***	.0378 *

*** significant, p<.0001; ** significant, p<.0005; * significant, p<.05

1-strongly disagree, 2-disagree, 3- somewhat disagree, 4- neutral, 5-somewhat agree, 6-agree, 7-strongly agree

According to the analysis, it is highly evident (with all p-values being 0) that the means for all statements (Q1_1 to Q1_7) are greater than 4. This indicates that students had a significantly positive view of the alternative approach. Furthermore, when focusing on the statements regarding students' comfort in mastering the alternative approach (Q1_1, Q1_3, and Q1_5), the analysis reveals that it is highly evident (with all p-values being less than 0.05) that the mean is greater than 5. This suggests that students were able to master the alternative approach relatively well. Notably, for the statement measuring the overall effectiveness of the alternative approach (Q1_7), it is highly evident that the mean is greater than 5.5. This indicates that students significantly agreed with its effectiveness.

The second part of the survey comprised of three multiple-choice questions (Q2, Q3, and Q4) comparing the textbook approach and the alternative approach. After Q2 and Q4, students were asked to provide their written perceptions of their choices in the form of open-ended questions (Q2_1 and Q4_1). The data collected from these questions was visually represented in Figure 4 using a barplot.

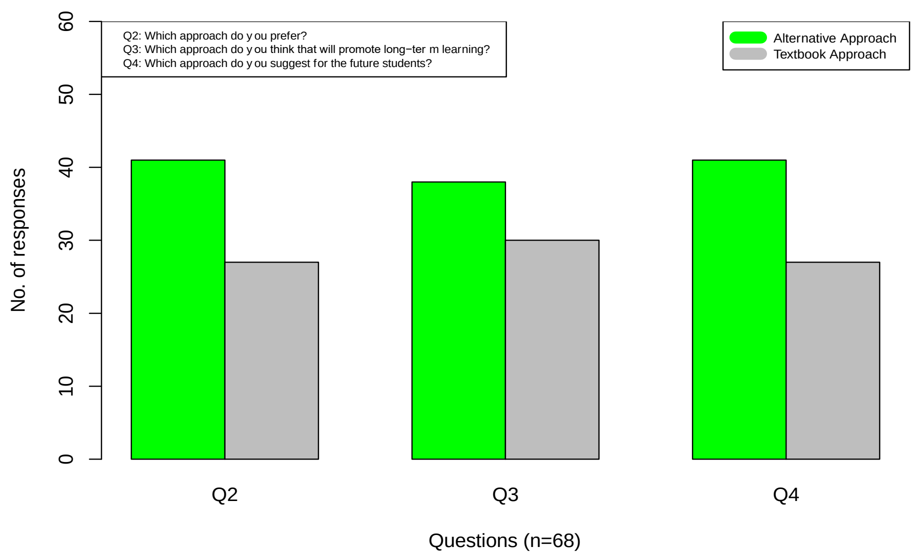


Figure 2: Students choices between "Textbook Approach" and "Alternative Approach"

From the observations made in Figure 4, it is evident that a larger number of students express a preference for the alternative approach. Furthermore, a higher percentage of students believe that the alternative approach facilitates long-term learning. Additionally, a greater number of students indicate their willingness to recommend the alternative approach to future linear algebra students.

To further substantiate the observations, a one-sample proportion test, specifically a large sample z-test, was conducted. The objective was to evaluate the population proportion (represented by "p") of students who chose the alternative approach. The hypothesis test was formulated as follows: $H_0: p = .5$ (null hypothesis) versus $H_a: p > .5$ (alternative hypothesis). The conditions required for the proportion test, np ≥ 10 and n(1-p) ≥ 10, were satisfied, making it a valid test.

Table 3: Analysis on students' choice between "Textbook Approach" and "Alternative Approach"

	Alternative Approach	Textbook Approach	$H_0: p = .5$ vs $H_a: p > .5$ p-value
Q2 - "prefer"	41/68 (60.3%)	27/68 (39.7%)	p=.0448 **
Q3 - "long-term learning"	38/68 (55.9%)	30/68 (44.1%)	p=.166
Q4 – "suggested for future students"	41/68 (60.3%)	27/68 (39.7%)	p=.0448 **

** significant, p <.05

Based on the analysis presented in Table 3, the obtained results indicate that there were notable differences among the responses. Specifically, the small p-value (p = .0488) suggests that a significantly higher proportion of students preferred the alternative approach and would recommend it to future linear algebra students. However, in contrast, the aspect of "long-term learning" did not show a significant difference between the two approaches (p = .166). It is important to note that these findings highlight the varying perspectives of students when evaluating the different aspects of the alternative approach.

Qualitative analysis and results of students' perceptions

After students indicated their preference in Q2, which asked them to choose between the two approaches, they were asked to provide written explanations in response to Q2_1: "Explain why you picked that choice from the previous question." Each response was carefully reviewed, and an inclusive analysis approach was employed to analyze the collected data.

Based on students' reflections, various reasons for preferring the alternative approach emerged. Some common ones include the perception that it is easier to understand, requires less logical reasoning, is easier to learn and apply, facilitates better retention of information, decreases the likelihood of errors, offers a quicker problem-solving process, provides a consistent method for solving any type of problem, reduces the possibility of making logical mistakes, feels more intuitive, enhances understanding and interpretation of questions, minimizes the need for memorization, and is deemed more reliable overall. These insights provide valuable insights into the specific benefits perceived by students in relation to the alternative approach. Some response examples to support the alternative approach are:

- *I pick the alternative method because it's the same procedure no matter the type of change of basis I perform. Textbook method requires a memorization of the chart for U, V inverse.*
- *I forgot how to do the textbook, and alternative way is much easier to remember and understand.*
- *I think the "alternative method" is a lot more straightforward and easier to understand compared to the "textbook method."*
- *This method allows for a simple and easy way to find the change of bases through linear systems we can solve. This is something we have done all semester, so I feel more comfortable with this method.*
- *It was more intuitive, it's much simpler and easier to retain.*
- *Easier to see how the coordinate vector affects the change of basis matrix.*
- *The alternative method seems like it has less possibility of logically messing up.*
- *It helps break down the problem into smaller parts.*
- *Better long-term learning*
- *I prefer the alternative method because I don't have to memorize which matrix I have to invert, and it also avoids matrix multiplication.*
- *The Textbook method is easy to mix up.*

- *I think the Textbook Method might have been easier if I had remembered the formula but the process for the Alternative Method is easier to undergo without memorization.*
- *It aligns more with our understanding of solving linear systems.*
- *More conceptual than just memorizing a formula like the textbook approach.*
- *This method makes more sense logically and easier to derive with previous knowledge.*
- *It's easier to remember and doesn't require as much logic.*
- *easier to understand with explanation of coordinate vector.*
- *you can visualize easier why the transformation works. it matches the way linear algebra concepts for taught using the e_i vectors.*

In contrast to the varied explanations supporting the alternative approach, students' perspectives on the textbook approach were more focused and limited. Their preference for the textbook approach mainly stemmed from a few key reasons: 1) difficulty in understanding the alternative approach, 2) perceiving fewer steps to calculate after their memorization, and 3) having a better understanding of the underlying logic in the textbook approach. It is worth noting that even among those who chose the textbook approach, they acknowledged the challenge of accurately memorizing formulas.

One possible explanation for some students' difficulty in grasping the alternative approach could be the sequencing of instruction, where the textbook approach was taught first and extensively practiced before introducing the alternative approach. It's important to consider that the survey was conducted shortly after the introduction of the alternative approach, which may have influenced students' responses. Some typical statements made by students who picked the textbook approach include:

- *I don't completely understand the logic behind the alternative method. if I did, I may be more comfortable using the alternative method.*
- *I can logically remember how to get to standard basis and the to other basis, so I never forget what order the vectors go.*
- *I understand where the formula comes from, making it easy to reproduce and adapt to the question.*
- *It's just a simpler method to implement but I did have some difficulty with the order of which matrix your supposed to find the inverse of*
- *I prefer the textbook method because it includes less steps so if I have the formula memorized it is harder to make mistakes than with the Alternative Method*
- *I prefer the textbook method since it helps me understand change of basis as a concept more easily and I feel like after memorizing it, it is harder to make mistakes with the textbook method since it involves less steps for computation.*
- *I still think it's more logical and there are fewer steps so it's harder to mess up the math. The only thing you need to know to do the textbook method is know how to change to the standard basis vector.*
- *It requires less calculations.*
- *I feel like the textbook method is more straightforward to understand, but if you don't completely memorize it, it's harder to implement than the alternative method which is more so of a logical progression.*

Like Q2, after students indicated their suggestion for the future linear algebra students(Q4), they were asked to provide written explanations in response to Q4_1: "Explain why you picked that choice". The students' choices in their suggestions were mostly consistent with their preference, However, more perception was gauged in Q4_1 with sample responses displayed below:

- *I think using the Alternative Method to solve is faster, more reliable, and easier. HOWEVER, I do think the Textbook method is better for LEARNING the content. Starting by multiplying by the COB to get the standard, then the inverse to get the new base, and then combining them makes for a much deeper understanding of how/why the content works. However, as far as actually solving these, I'm always going to use the Alternative Method from now on*
- *What may work for me may not work for another person, so I feel like both should be taught.*

- *If they find memorizing formulas more difficult the Alternative Method would be easier*
- *Textbook method is more logical and mathematically backed. The alternative method is kind of a shortcut.*
- *I think that it is easy to memorize the alternative method, but the textbook method will stay in my head. Both should be understood as they develop a better understanding for change of basis and can be a better approach to different problems.*
- *The alternative approach is easier to comprehend. However, to understand what a basis is and how it works, I think the textbook method is best taught first then the alternate method is given as a "shorter" way.*
- *If they find memorizing formulas more difficult the Alternative Method would be easier*
- *I think both are good, the textbook one might make more sense conceptually, but the alternative method is more intuitive and easier to remember.*

When considering the statistically supported evidence, it becomes clear that the alternative approach is preferred by more students and is recommended for future linear algebra students. However, it is important to acknowledge that both methods have their own strengths and weaknesses.

For instance, despite the previously mentioned disadvantages of the textbook approach, it does offer the advantage of providing valuable insights into the structural changes that occur when transitioning from one basis to another. By demonstrating how the textbook approach works, students can gain a deeper understanding of the content, as one student explained: "I do think the Textbook method is better for LEARNING the content. Starting by multiplying by the CBM to get the standard, then the inverse to get the new base, and then combining them makes for a much deeper understanding of how/why the content works."

However, the effectiveness of the textbook method heavily relies on the instructor's approach. If instructors solely rely on theoretical proofs without engaging explanations, it can become monotonous and hinder student comprehension, particularly for engineering students. On the other hand, the alternative approach has its advantages and is less dependent on instructor explanations as the procedure is fixed. However, it does not reveal the observation that the standard coordinate system connects when converting between two nonstandard coordinate systems. This highlights the trade-offs and limitations of each approach.

Conclusion

The determination of the change of basis matrix (CBM) is a crucial topic in linear algebra education. Traditionally, this topic has been taught using textbook approaches, which present certain challenges. These challenges include explanations that heavily rely on mastery of complex concepts, making it difficult for engineering students to understand the associated formulas. Additionally, the formulas themselves can be inherently complex, leading students to struggle with memorization and comprehension of the sub-indexes, orders, and symbols involved. Recognizing the recurring mistakes made by students when applying these long-standing textbook methodologies, we have identified an alternative approach that can enhance students' performance in this area.

This study aimed to introduce and investigate the effectiveness of an alternative approach in teaching the determination of the change of basis matrix (CBM). The alternative approach is grounded in the concept of determining the linear transformation matrix (LTM) for LT: $R^n \to R^m$, a standard concept that students are typically required to understand and can grasp well in a linear algebra class. It relies on the comprehension of coordinate vectors and the proficiency in solving linear systems, eliminating the need to master complex concepts such as composition of change of basis or memorization. As a result, it alleviates the difficulty associated with the problem-solving process.

The analysis of the survey data indicates that students have a significantly positive perception of the alternative approach. They expressed comfort in mastering the alternative approach, demonstrated good proficiency in understanding it, and believed it to be effective overall. Moreover, a significant majority of students preferred the alternative approach and recommended it for future linear algebra students. These preferences were supported by various advantages identified by the students themselves. These advantages include easier comprehension, reduced reliance on logical reasoning, simplified learning and application, improved retention of information, decreased likelihood of errors, faster problem-solving process, consistent method for any problem type, minimized probability of logical mistakes, intuitive understanding, enhanced interpretation of questions, reduced need for memorization, and overall perceived reliability. Furthermore, the implementation of the alternative approach has resulted in significant improvements in student performance on these topics.

By the study, the alternative approach proves to be an effective teaching method for enhancing students' performance. However, it is important to acknowledge that it does have limitations, as discussed previously. Therefore, from the author’s perspective, Linear algebra instructors are advised to introduce the alternative method when time is limited. However, if time permits, it is recommended to introduce the alternative method first, supplemented by the textbook method, to maximize the benefits for students' learning.

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