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# Developing Active Learning of Linear Algebra in Engineering by Incorporating MATLAB and Autograder

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# 1 Abstract

The purpose of this paper is to redesign a traditional linear algebra course that focuses mainly on theoretical concepts without any numerical application, in order to enhance the learning experience for students. This was achieved by conducting a comprehensive study of the students' needs, contents covered, pedagogical approaches used at peer institutions, and existing literature published by experts in the field.

The redesigned course incorporates the following changes to make it more dynamic, computational, and applied:

- Significant integration of active learning through in-class group worksheets that embed MATLAB practice problems.
- Devlopement of MATLAB live scripts prior to class to visualize abstract concepts when teaching and compute more complex problems.
- Inclusion of coding core linear algebra concepts as a class task.
- Addition of application projects to the curriculum.
- Adoption of the MATLAB Autograder to grade student work.

As a result of these changes, the course is now more interactive and application-focused, leading to a more positive impact on students' learning experience, as reported by the majority of students.

# 2 Introduction

The importance and utility of linear algebra have continued to increase, and a solid understanding of linear algebra has become essential in many fields. Linear algebra techniques are used to solve problems in a wide range of disciplines, including engineering, computer science, operations research, economics, and statistics. The knowledge and application of linear algebra is particularly emphasized by various ABET program criteria, making it a required skill.

To address the need for improving the undergraduate linear algebra curriculum, the Linear Algebra Curriculum Study Group (LACSG) was established in January 1990[1] . This group had a significant impact on the design of linear algebra textbooks and courses. Meanwhile, due to the unique characteristics of the course, various teaching experiments and approaches have been implemented to overcome obstacles encountered by students when learning linear algebra[2][3][18][13][18]. Many experts in the field have proposed pedagogical innovations for teaching this course. These innovations are aimed at improving the effectiveness of teaching and enhancing students' learning experience[9][10][2][11].

It is widely recognized that traditional learning approaches are becoming less effective [14], particularly for Linear Algebra. Researchers have pointed out that the power of linear algebra to solve problems of greater complexity can be significantly enhanced through the use of hardware and software[2][9][1]. The Linear Algebra Curriculum Study Group (LACSG) and other researchers have also suggested that the use of computers for homework and projects can reinforce concepts from lectures and enable the solution of realistic applied problems[11].

The integration of technology in the teaching and learning of linear algebra can have a positive impact on students' understanding and application of the subject matter. By leveraging the computational power of computers, students can explore complex concepts and solve more challenging problems. This approach also provides opportunities for hands-on learning and promotes active engagement in the learning process. As such, the incorporation of hardware and software in the teaching of linear algebra has become increasingly important, as it enhances students' problem-solving skills and prepares them for real-world applications. This approach not only improves the effectiveness of learning but also better aligns with the evolving demands of today's job market.

At the University of Virginia's School of Engineering & Applied Science (SEAS), the course APMA 3080 - Linear Algebra is a critical component of the engineering curriculum. However, in the past, the course was primarily theoretical, lacking any significant numerical component. We refer to this past traditional course as PTC in this paper. We identified several shortcomings with the PTC course, including but not limited to: firstly, engineering students are expected to apply the knowledge they acquire in school to real-world scenarios, yet PTC lacked any application component. Linear algebra has numerous real-world applications, and engineering students require a practical understanding of how to apply the subject matter. Secondly, coding plays a significant role in engineering, and MATLAB is an ideal tool for working with linear algebra, yet no numerical component was adapted in PTC. Moreover, teaching the course in a passive traditional way has consistently been challenging, especially for a subject like linear algebra that covers various abstract concepts. Students often get easily distracted in class, and they need to spend a significant amount of time outside of class to understand the material thoroughly.

To address the issues identified in the traditional PTC course, the author embarked on a redesign effort aimed at achieving several objectives. Firstly, the goal was to make the learning experience more active, interesting, motivated, and efficient. Secondly, the aim was to better illustrate the power of linear algebra to explain fundamental principles and simplify calculations in various fields, including engineering, computer science, mathematics, physics, biology, economics, and statistics. Thirdly, the focus was on better communicating the importance of linear algebra in the applied field, reflecting it as a scientific tool. Lastly, the objective was to empower students' abilities to solve more complicated and applicable problems in the real world. This paper's primary focus is on the redesign effort, which incorporates MATLAB and introduces active learning into the course, while still covering all the core topics in any basic linear algebra class. This approach makes the course more computational, dynamic, and applied, leading to improved students' understanding of the material and faster learning efficiency. This redesigned course structure will be referred to as the course with active learning and MATLAB (CALM) in this paper.

There is a significant body of literature on pedagogical approaches to teaching college-level linear algebra [2] [3][7][18][8][9][10][2][11] [13][19][20], some of which inspired our redesign, which we will discuss in more detail later. However, the author has not found any existing method that incorporates as many innovations as our redesigned course. Most of the literature focuses on one or two aspects, and the closest approach to our redesign was conducted by professors from UIUC[2], who incorporated Python into the course structure. Their approach involved two lectures and one lab per week, with the lab component using interactive activities and an autograder. However, the lecture component remained traditional. In contrast, our redesign incorporates MATLAB and an autograder, as well as active learning during each class meeting. Furthermore, we ask students to code the core linear algebra concepts, which the author has not observed anywhere else. It should be noted that our redesign and the UIUC approach[2] were developed independently, with both incorporating a numerical component and an autograder, but using different programming languages (MATLAB and Python, respectively).

CALM incorporates several innovative approaches to teaching linear algebra, including: (i) active learning is significantly integrated into each class through in-class group worksheets that incorporate MATLAB to solve more complex problems. (ii)instructors utilize MATLAB live scripts to visualize abstract concepts and foster an interactive learning environment when teaching, (iii) students are tasked with coding core linear algebra concepts, providing them with a different perspective and deepening their understanding of the material, (iv) application projects are added to demonstrate to students the interesting and applicable nature of linear algebra, and (v) MATLAB Autograder is adapted to grade students' work, providing immediate feedback and reducing the burden of grading for instructors.

# 3 Content Revision

Starting from Spring 2021, the author embarked on an investigation into how to improve the course. The first step was to examine the course content. To achieve this, the author reviewed existing research papers and textbooks (eg. [21][22][23][24][25]) while also actively seeking feedback from various engineering programs at SEAS from UVA regarding the topics required for their students from this course. Additionally, the author carefully examined the list of topics and application projects delivered to engineering students from peer institutions, such as Stanford University, MIT, UPENN, UIUC, UMICH, UC Berkeley, TAMU, and VT. Numerous valuable discoveries were made during this process. In addition to the wealth of extensive research on various teaching methods, a few findings related to the peer institutions are listed below, but not exclusively.

- More and more programs or instructors, including SEAS at UVA, tend to include a numerical component in linear algebra. Six out of eight universities have utilized software to a certain to enhance the learning experience. From such six institutes, the lab is either a component in the course(homework) or a second version of the same course with a lab is provided.
- Each linear algebra class covers the core topics suggested by LACSG. But engineering linear algebra class usually does not include detailed intruction about generalized vector spaces. Instead, tit just covers a simple introduction of such topcis.
- There are a few common linear algebra projects if the class involves the component of application, e.g., Markov Chain, Least Square Regression, and Image Compression by SVD. All other application projects vary from school to school, depending on students' needs and interests.
- Four out of eight universities have discussion sessions in addition to the regular three traditional lectures per week.

Bases on all the findings the author investigated, along with the objectives of making the learning more active, efficient and applicable, the author carefully reconstructed the course contents, which is displayed in Figure 1.

| | | |
|---|---|---|
| **Systems of Linear Equations (4 hours)** | Linear systems, Elementary matrices, echelon/reduced echelon form, Gaussian/Gauss-Jordan Elimination | MATLAB: (i) code core concepts. (ii) Project 1: Traffic Flow or Loop Currents |
| **Vector Spaces-P1 (4 hours)** | 1, Vectors | MATLAB: code core concepts. |
| | 2, Linear combination, Span | |
| | 3, Linear independence/dependence | |
| **Matrices (5 hours)** | 1, Linear/matrix transformations in R^n | MATLAB: (i) code core concepts. (ii) Project 2: Encrypting and Decoding |
| | 2, Matrix Algebra | |
| | 3, Inverses | |
| **Determinant (4 hours)** | 1, 2 by 2 & 3 by 3 determinant, cofactor expansion | MATLAB: (i)code core concepts. (ii) Project 3: LU Facorizationa and Steady State Heat Flow Problem |
| | 2, Properties of the determinant, eg, det(AB)=det(A) det(B) | |
| | 3, Introduction of Cramer's Rule | |
| **Vector Spaces-P2 (7hours)** | 1, Subspaces of R^n, | MATLAB: (i)code core concepts. (ii) Project 4: Graphs and networks & Incident Matrices |
| | 2, Basis and dimension | |
| | 3, Row and column spaces, Null space, rank, nullity-rank thm | |
| | 4, Coordinate vector, (Opitional) simple introduction of Change of basis | |
| **Eigenvalues and eigenvectors(7 hours)** | 1, Eigenvalues, eigenvectors, eigenspaces, Identify some of the characteristice polynomia's coefficients (trace, determinant) | MATLAB: (i) code core concepts. (ii) Project 5:Google Page Rank with embeddied Markov Chains |
| | 2, Similarity/Diagonalization | |
| | 3, Symmetric matrices, orthogonal diagonalizatoin, quadratic forms (teach after Gram-Schmidt) | |
| **Orthogonality(7 hours)** | 1, Dot product/inner product, length and orthogonality | MATLAB: (i) code core concepts. (ii) Project 6: SVD/ image compression by SVD/SVD applied to Principle Component Analysis. (iii) Project 7: Least Square & data(curve)-fitting |
| | 2, orthogonal/orthonormal sets and bases, orthogonal matrices. | |
| | 3, Projection, Gram-Schmidt Process | |
| | 4, Least Squares Regression with application to data-fitting | |
| **Abstract/generalized vector spaces (3 hours)** | Introduction of Abstract/generalized vector spaces, inner product spaces | MATLAB Project 8: Error-Detecting and Error-Correcting Codes |
| **total ~38-41 hours** | | |

Figure 1: Proposed schedule in Spring 2021

From the perspective of content, there are several main differences when comparing with PTC. Firstly, we removed the detailed instruction on generalized vector spaces(GVS), which took around three weeks in PTC to cover. Instead, we distribute one week to teach a simple introduction. The detailed topics about GVS usually are taught for math majors in a second or an advanced version of a linear algebra course. Considering our audience are engineering students, it is evident that numerical applications are preferred. The discoveries from the mentioned peer institutes also validated such revision. Secondly, we add numerical components, which are not included in PTC . There are four parts for the newly added numerical component: MATLAB live script for instructors to teach, MATLAB practice problems in group worksheet during each class meeting, coding basic concepts in MATLAB Grader, and MATLAB application projects in MATLAB Grader. By writing MATLAB programs, students have to imagine the actions carried out by the computer in response to their commands and anticipate the outcomes of these actions. These mental activities of imagination and anticipation involve mental manipulations of vectors and matrices that constitute an essential component of the linear algebra environment. Lastly, we added topics of orthogonal diagonalization for symmetric matrices and quadratic forms as core concepts instead of optional choices as in PTC. Such skills are needed for many engineering applications and are included as core concepts by LACSG. Their importance can be seen via the feedback from SEAS at UVA and some peer institutesdai as well.

# 4 Implementation

After knowing the topics that needed to be covered, in Fall 2021, the author started to develop the teaching materials according to the suggested schedule, and CALM was delivered to students as the first time in Spring 2022. The typical way to teach CALM is described as following: in a class of 50 minutes, the instructor teach the core concepts and use the pre-written MATLAB lives script to enrich the explanation when necessary. This part takes around 20 to 30 minutes. Then students work as small groups in class to finish group worksheet handed out to them at the beginning of the class, and this part takes the rest of the class. When finishing a block, the instructor discuss briefly about coding core concepts and application projects in MATLAB from that block and student will finish them after the class.

## 4.1 Lecture: Interactive MATLAB live script is incorporated

Linear Algebra is extensively abstract. In a typical traditional class, students are easily get confused by various concepts since they could not actually 'see' them or connect them with life, hence it is very common for them to get distracted in the classroom settings. Some Students might need to spend much more time after the class to master the concepts, while some others might struggle with them for a long time. In addition, it happens very often that students keep asking 'what they are' even they can solve problems related with such concepts mathematically. In CALM, MATLAB live script is written prior to the class by the author to visualize such abstract concepts, so that students can actually 'see what they are'. Consequently, students could obtain a quick mastery of the material just in class. Visualizing thoretical concepts not only catches students' attentions closely but also makes the learning much efficiently[15], since human brain is built to get concepts much faster if connected with images.

### 4.1.1 Example 1

Two core concepts from Linear Algebra are 'linear combination' and 'span' of a set of vectors. In PTC, some students might be able to solve problems related with them very mathematically on the paper, but it happens so often that they don't actually know what they are. And lots of students consistently got lost about the span of two linearly independent vectors in $\mathbf{R}^3$. Many of them said that such span should be $\mathbf{R}^2$, which is incorrect. Prior to the class, the author developed MATLAB live script shown as in Figure 3. Such design focuses on student -centered active learning. It features (i) introducing the concepts of "linear combination" and "span" (ii) the instructor visualize such concepts by rotate viewing the graphs (iii) students revise the given codes to explore more (iv)students reflect what they have observed after each step and ultimately get some conclusions themselves.

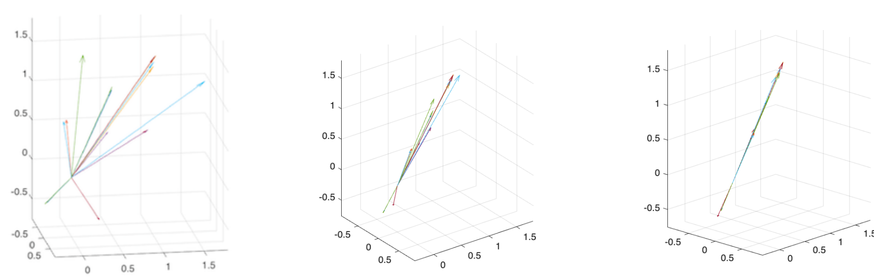


Figure 2: Visualize that the set of two linearly independent vectors in $R^3$ actually spans a plane by rotate viewing the graph obtained in the example from Figure 3

## 2. Visualize linear combinations and spans

$\mathbf{u}_i \in \mathbf{R}^n$, a **linear combination** of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$ for any choice of $c_1, c_2, \cdots, c_n \in \mathbf{R}$.

A **span** of the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is the collection of all combinations of them, i.e.

$$span\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\} = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n | c_i \in \mathbf{R}\}$$

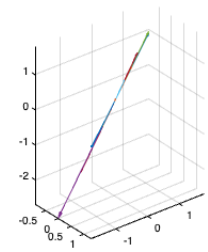
**Student Reflect:** Could you tell exacatly the difference between a span and a linear combination?

### 3D Case

**Example**. Given two vectors **u**=[1 0 2], **v**=[-1 2 1];

- in the same graph, plot **u**, **v**, then plot some linear combinations of **u**, **v**: $a\mathbf{u} + b\mathbf{v} = [a-b \quad 2b \quad 2a+b], a, b \in \mathbf{R}$

```
quiver3(0,0,0, 1, 0,2)
axis equal
hold on
quiver3(0,0,0, -1, 2, 1)
hold on
% now plot some linear combinations
a=  0.299                ; b= 0.2915              ;
quiver3(0,0,0, a-b, 2*b, 2*a+b)
%hold off % if you want to see all vectors with different values for a, b, comment this!
```

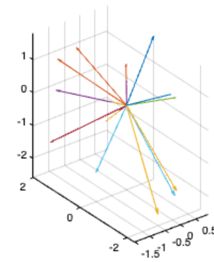


**Student Reflect:** In the above example, what can you obeserve? If a, b take up all values in **R**, what can you conclude?

**Student Exercise**. Based on the above two vectors, a third vecor is given as **w**=[0 1 -1], which is not a linear combination of **u, v**;

- in the same egraph, plot **u**, **v**, **w**, then plot some linear combinations of **u**, **v**, **w**: $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = [a-b \quad 2b+c \quad 2a+b-c], a, b, c \in \mathbf{R}$

```
% student works below by modifying the code above
quiver3(0,0,0, 1, 0,2)
axis equal
hold on
quiver3(0,0,0, -1, 2, 1)
hold on
quiver3(0,0,0, 0, 1,-1);
a= -0.1095               ; b= 0.7505              ;c= 0.7495              ;
hold on
quiver3(0,0,0, a-b, 2*b+c, 2*a+b-c);
%hold off›
```



**Student Reflect:** In the above example, what can you obeserve? If a, b , c take up all values in **R**, what can you conclude?

Figure 3: Interative MATLAB live script to master the concepts of 'linear combination' and 'span' by visualizing them

For the example in Figure 3, the instructor randomly changeed values of $a$ and $b$ to generate some random linear combinations of two given vectors $\overrightarrow{u}, \overrightarrow{v} \in \mathbf{R}^3$. From here, students could see that a linear combination is actually a vetcor by putting some coefficients before each vector in the sum. Then the instructor rotate showed the graph and ask student what they have observed. Student reflected that all such linear combinations residue inside one plane in $\mathbf{R}^3$ instead of equal to $\mathbf{R}^2$. Then students modified the codes in the interactive box to randomly plot some linear combinations of three linearly independent vectors, which is the exercise in Figure 3. By rotate viewing the graph they obtained, they observed that the span is not a plane any more. All the combinations residue in all possible directions in $\mathbf{R}^3$.

Then students reflected that: three linearly independent vectors span the whole $\mathbf{R}^3$.

### 4.1.2 Example 2

Another important concept in this class is 'subspace'. In order to determine whether a given subset $S \subset \mathbf{R}^n$ is a subspace, three criterion should be checked: (i) $\overrightarrow{0} \in S$. (ii) $\forall \overrightarrow{u}, \overrightarrow{v} \in S$, then $\overrightarrow{u} + \overrightarrow{v} \in S$. (iii) $\forall r \in \mathbf{R}$ and $\forall \overrightarrow{u} \in S$, then $r\overrightarrow{u} \in S$. Students always get confused about such three criterion even with multiple repeated explanation. In PTC, I was asked numerous times for the real meaning of such criterion. Some students could follow examples step by step, but it is very mechanically without a deep understanding. The author used MATLAB scripts as a tool to help with illustrating this concept. For the example in Figure 4, the author showed three criterion one by one, and students immediately reflected that they understood what do the three criterion mean and this set $S$ did not pass any rule, hence it is not a subspace.

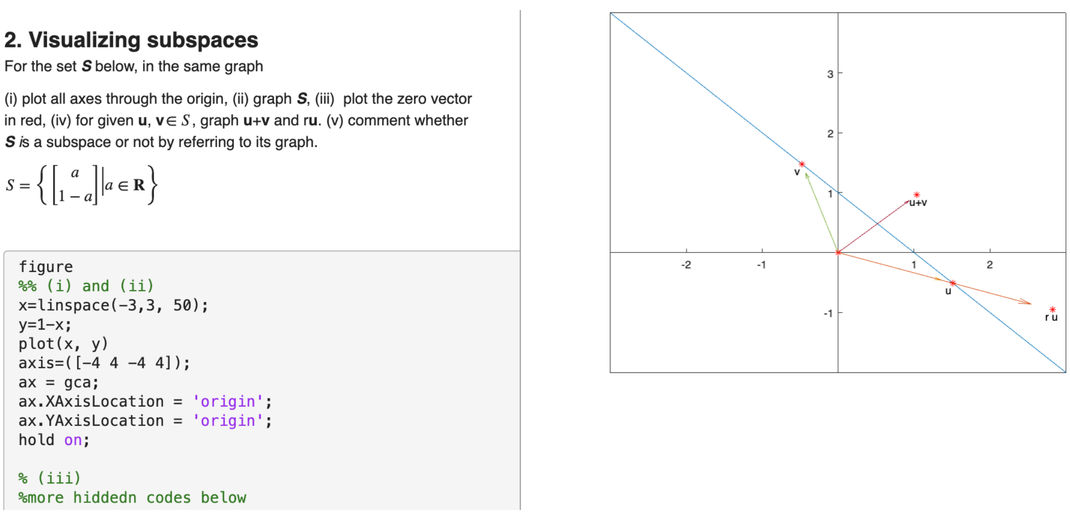



Figure 4: Visualize a non-subspace

## 4.2 Active learning by doing in-class group worksheets with MATLAB problems embedded

Collaborative learning is a proven method of teaching that has gained popularity among instructors of STEM courses in higher education. Research has shown that involving students in collaborative activities results in better retention, improved learning outcomes, and increased persistence 16] [11] [14]. Instructors recognize the advantages and efficiency of active learning and are eager to introduce more such activities in the classroom. Studies have also demonstrated that inquiry-based learning activities can sometimes be more effective in promoting learning among students[20]. However, readily available off-the-shelf activities for a particular topic or course are hard to come by.

To create a more dynamic learning environment, a worksheet is distributed to each student group during class. Working together on the worksheets not only provides students with more practical learning opportunities compared to traditional lectures but also allows them to communicate their thought process with their peers - a skill highly valued in the engineering field. Instead of spoon-feeding information, the worksheets are carefully designed to (i) tackle more complex problems with greater magnitude, which cannot be solved by manual computation, (ii) reinforce the concepts taught in lectures and (iii) encourage students to explore and discover knowledge themselves, As soon as the groups start working on the worksheet, the classroom becomes alive with various questions and discussions. With around 50 students in the classroom, having assistants is crucial. The center of Applied Mathematics(APMA) from UVA have played a dominant role to train undergraduate teaching assistants(UTA) [7][8], and UTAs are widely hired by APMA faculty to help in the classroom. When students work in groups, a trained UTA and the instructor provide support, responding to any questions raised by students. This creates a highly interactive environment, keeping us occupied with various student queries, which is an indication that students are actively engaging in the learning process. This aspect of the class is usually well-received by students.

5. Download .m file *Checkspan* from Collab, determine whether $\mathbf{b} \in span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 7 \\ -10 \\ 4 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 5 \\ -5 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \\ -5 \\ 1 \end{bmatrix}$. The instruction to use *Checkspan* is given in the same .m file, or please watch this video to learn how to write and run a function in MATLAB.

Figure 5: An example from the worksheet to enable students to compute more complex linear algebra problems with great magnitude by MATLAB

There may be a few students in our class who are unfamiliar with MATLAB, but this should not hinder their ability to participate fully. To support their engagement, we have embedded links to helpful videos or documentation within the worksheet, which provide guidance on the necessary MATLAB commands. Rest assured that the commands we use in class are intuitive and uncomplicated, and most students tend to grasp them with ease.

(b) Now consider that there are 100 sets of vectors we need to check independence, it's obvious we don't want to repeat solving linear systems for 100 times by hands. Instead, due to the core reason to be linearly independent, we can write a MATLAB function, like a 'machine', with input to the set of vectors and the output be the conclusion for its independence. Please download *CheckLide.m* from collab and go through the code line by line for the case $\left\{\begin{bmatrix}1\\2\\0\\4\end{bmatrix}, \begin{bmatrix}1\\4\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\4\\-12\end{bmatrix}\right\}$

(a) A=

(b) What is the reduced echelon form of A , namely R=

(c) How many pivot columns in the reduced echelon form, namely *length(p)*=

(d) Think about the meaning of lines 14-15, and explain why the set is linearly independent if '*length(p)==size(A,2)*'

(e) Is the set linearly independent by calling this function?

1. In the previous worksheet, you solved some questions by hands involving the following matrices

(i) $A = \begin{bmatrix} 2 & 1 & 0 \\ 6 & -3 & -1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -4 \\ 1 & 3 & -7 \end{bmatrix}$ (iii)$A = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 1 & 4 & 2 & 6 \\ 0 & 3 & 3 & 3 \end{bmatrix}$.

In MATLAB, please use *rref* to find RREF for each matrix above.

2. Now can you determine whether the columns of the following matrices span $\mathbf{R}^3$ quickly?

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & -7 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$

3. Keep the above two questions in your mind, please compare the number of pivot columns in REF or RREF with $n$ in $\mathbf{R^n}$(eg. $n = 3$ in $\mathbf{R^3}$),

if the columns of the matrix with $n$ rows don't span $\mathbf{R^n}$, what is the relation between the number of pivot columns in REF(or RREF ) and $n$? How about if the columns of the matrix span $\mathbf{R^n}$?

Figure 6: Left: A worksheet example to reinforce concepts in MATLAB. Right: A worksheet example to explore more knowledge students themselves by using MATLAB as a tool

## 4.3 Code core linear algebra concepts in MATLAB with autograder incorporated

The coding component of our course is highly innovative. While some may question the need for coding in a linear algebra course, the benefits are numerous. As a self-motivated learner, I have found that only the courses where I wrote code deeply ingrained the material in my mind, even after many years. Coding reinforces long-term learning and helps to deepen a student's understanding of various concepts in linear algebra. Additionally, MATLAB and linear algebra are a perfect match, allowing students to explore a wider range of concepts than what would be possible through traditional methods alone. The coding component provides a different perspective for learning linear algebra, which students may have never considered before.

To facilitate this coding component, we have designed it using MATLAB Grader, an autograder that provides immediate feedback to students, which is incredibly valuable. Below is an example of the type of coding challenge we give students: write code to determine the number of solutions for any given linear system.

#### 4.3.1 Example

**Step1: Discuss the values to write codes**

Let us recall the process of determining the number of solutions for a linear system by analyzing its echelon form (REF) or reduced echelon form (RREF). To do so, we perform a series of elementary row operations on the augmented matrix of the system until we obtain its REF or RREF. Then, we examine the resulting form from two main perspectives: (i) Are there any false statements, such as 0=1, which make the system inconsistent? (ii) Are there any free variables, in case there are no false equations?

By answering these questions, we can determine the number of solutions for the linear system. However, when faced with thousands of large linear systems, it is not feasible to examine each one individually. Is there a more efficient way to solve such a problem? Are there any general criteria that apply to all linear systems?

For example, imagine that Mathworks or another company focused on creating online tools for solving linear algebra problems hires you as a software engineer. One of your primary tasks is to develop a function to determine the number of solutions for any given linear system, especially large systems. How would you approach this task?

Clearly, such a complex task cannot be solved through manual computation alone. However, by writing just a few lines of code, students can solve this problem for any linear system, making the task both efficient and applicable in practice.

It is crucial to have these discussions, as they motivate students to complete the work and provide them with a fresh perspective on the subject's value.

**Step2: Set up the problem in MATLAB Grader**

Please refer to the left-hand side of Figure 7. The problem setup is divided into three sections. The first section provides clear instructions on the task that students need to complete. The second section, known as the "Pretest", includes examples that are simple enough for students to solve by hand. This section serves the purpose of allowing students to check their codes without penalty before submitting their work for actual grading. The third section includes testing problems that typically involve large dimensions that cannot be solved on paper, but can be efficiently solved using MATLAB.

**Step3: Develop the criterion and feedback in MATLAB Grader**

Please refer to the right side of Figure 7. Students utilize the provided template to complete their code. Before submission, they can safely test their code by running the "Code to call your function" section to ensure it is functioning correctly. Upon submission, they receive prompt feedback on specific areas that require correction. This iterative process requires students to continually refresh their understanding of the concepts and ultimately reinforces their mastery of the material.

Throughout the course, our primary objective is to enhance students' comprehension and practical application of linear algebra, without getting bogged down by the intricacies of MATLAB syntax. To achieve this, we furnish students with templates containing MATLAB commands that are not related to linear algebra, and offer useful clues on pertinent MATLAB functions that can be utilized in problem-solving. This framework empowers students to concentrate on scrutinizing linear algebra principles embedded within the given code.

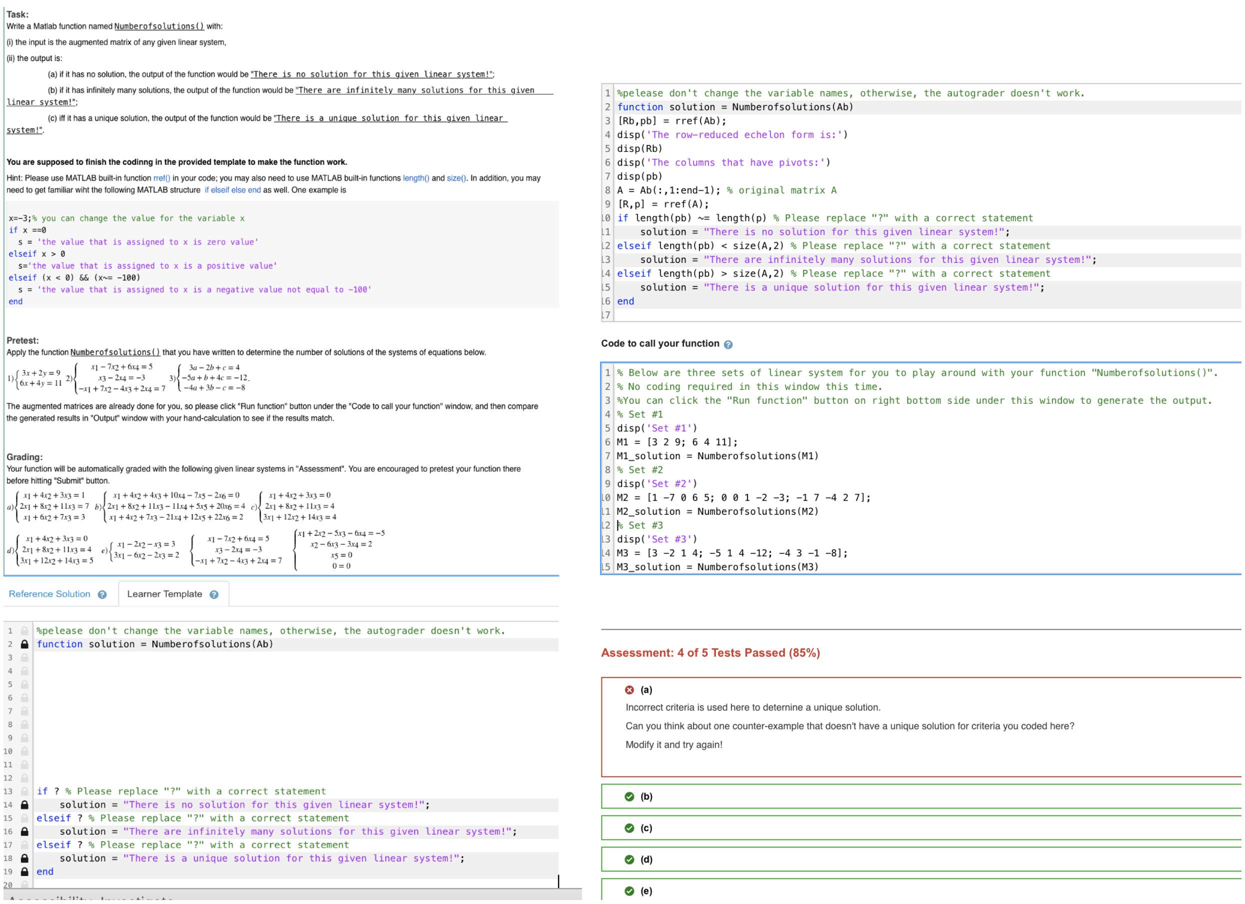



Figure 7: Left: An example to reinforce core concepts in MATLAB by writing codes. Right: The corresponding student's work which get autograded and immediate feedback

### 4.4 Establish application projects in MATLAB Grader

It is widely recognized that linear algebra plays a crucial role in numerous fields. Whenever you take a digital photo with your phone, apply filters in Photoshop, play a video game or watch a movie with digital effects, perform a web search, or make a phone call, you are utilizing technologies that rely on linear algebra concepts. From graphics, image processing, and cryptography to machine learning, computer vision, optimization, graph algorithms, quantum computation, computational biology, information retrieval, and web search, linear algebra has diverse and essential applications.

As our audience consists of engineering students, the practical application of linear algebra is particularly relevant. Regrettably, the previous course did not emphasize this aspect. As an instructor who teaches this course each semester, I have frequently been asked by students why Linear Algebra is necessary. Some students could not perceive the value of taking the class beyond meeting degree requirements, which hindered their motivation to explore the subject's beauty. To address this issue, the author introduced specific application projects as a component of the course through MATLAB Grader.

The project setups are similar to the "writing codes about core concepts of linear algebra" component, except that the problems focus on real-world applications instead of the core concepts themselves. This approach provides students with concrete examples of how linear algebra is utilized and enhances their understanding and motivation to master the subject.

Setting up the 'Core Linear Algebra' component and the 'Application Project' component in MATLAB Grader not only assists students in their learning but also significantly reduces the grading tasks for instructors and TAs.

## 5 Assessment

### 5.1 Survey

A survey was conducted in Spring 2022, we asked a few quesitons and some random written comments from students are quoted.

*Q: Did the projects help you to apply linear algebra to solve real world problems?*
122/166 (76.5 %) strongly agreed on that the projects empower them to apply the knowledge to real situations.

*Q: Please indicate how the component of application projects impact your learning if you agree with the obove question*

Student1: The applications were very useful and helped me strengthen my Matlab skills and see the benefits of linear algebra in real life.

Student2: The projects demonstrated how the concepts we learned can be applied practically to large-scale real world problems by basing the tasks around these sort of problems.

Student3: They made you think about the content in a different, deeper, way.

Student4: The code walked me through setting it up for the real world things so I never worried about how it would apply, what it did do is that it made me think about how the real world can be described using mathematical concepts which was very eye opening.

Student5: Because it made me approach the problems in a way that I didn't imagine at the beginning.

Student6: I think the projects had a good mix of challenge yet enough guidance so that it wasn't too difficult. I learned a lot about how we can apply linear algebra topics in the real world which I am grateful for

*Q: If there is any, please indicate how the component of MATLAB live scripts and the component of 'code core linear algebra concept in MATLAB' impact your learning, or indicate why not.*

student1: It made me think about the topics from class in a different way and more in depth

student2: It was a refresher of the material we had just learned and helped me see the concepts from a different perspective.

student3: It forced me to think about the concepts learned in class not just as numbers but as a coded representation of the concepts that really drove me to understand the concepts on more than a practical level

student6: Having access to MATLAB allowed a greater understanding of the material as MATLAB made it easy to mess around with matrices and become more comfortable with them

student4: I feel like the labs challenged me to really understand and think about the meaning of certain concepts and all of the properties of those concepts, especially with independence.

student5: It required knowledge of what we covered in class to complete the code so I had to go back and review the material. Very helpful.

student6: They helped me apply the manual process we learned in class to a coding algorithm to perform a similar process. Looking at each concept from both angles helped to facilitate a deeper understanding.

The majority students reflected positive impact on their learning from CALM. Despite the benefits of such redesign, we have received feedback from some students who feel that debugging the codes takes up too much of their time, leaving less time to focus on understanding the underlying linear algebra concepts. To address this concern, we provide a template that includes the non-linear algebraic part of the code, reducing the time spent on debugging. We also offer supplementary videos that help students become more familiar with MATLAB and hold daily workshops run by graduate students who can answer all types of questions.

Since receiving significant positive feedback from students in the Spring 2022 semester, we replace all sections of APMA 3080 by the newly redesigned version in Fall 2022 semester. While we no longer conduct formal surveys each semester, we have observed a decrease in the number of students who express concerns about learning MATLAB. It is possible that some students may have initially been resistant to the new technology, assuming the class would be taught in a traditional manner, and stopped considering its potential benefits. However, our approach has proven to be successful in improving students' understanding and application of linear algebra concepts. As students became more comfortable with the tool and the class format, we have seen an increase in overall student satisfaction with the course. Now, in our third semester of implementing these changes, we have received no complaints about MATLAB.

## 5.2 Tests

In PTC, it is common for students to have an average around 80/100. In CLAM, students had comparable average around 80/100 as well. Although the average test scores have remained relatively unchanged, we have implemented more challenging questions in the tests. We included multiple choice problems in the exam. Each multiple-choice question has either one or two correct answers. Students have reported finding this section particularly difficult, as they must determine the number of correct choices themselves. This design encourages students to thoroughly understand the concepts in order to score well, resulting in a stronger understanding of the material. In other words, students mastered skills better than PTC.

In addition to the more challenging tests, the CALM approach has also resulted in more efficient learning for students, as they are less distracted during class time. This has been reflected in the course evaluations, where students reported spending an around 2 hours less time outside the classroom while achieving comparable or even better test performance.

This increased learning efficiency is significant, as it means learners can achieve competence in less time, which is especially beneficial for college students who often juggle multiple classes and social organizations. With the amount of information presented in each class, it is easy for students to become overwhelmed. Therefore, the CALM approach not only helps students to achieve a stronger understanding of linear algebra but also alleviates some of the stress associated with college-level courses.

Moving forward, we plan to further refine the CALM approach to optimize learning outcomes. This includes exploring different pedagogical techniques, incorporating more real-world problems and applications, and providing additional resources and support to help students succeed. By doing so, we aim to continue improving the quality and effectiveness of linear algebra education at UVA and beyond.

# 6 Conclusion and Future Plan

Our project was motivated by the need to update and modernize the traditional engineering linear algebra course at UVA. We recognized that the past course relied heavily on abstract theory and lacked real-world applications, which could lead to a lack of interest and engagement from students. Our team aimed to address these shortcomings and make the course more relevant and practical for students.

To achieve this goals, we implemented a novel approach called course's active learning with MATLAB (CALM). This approach incorporated multiple computational components, including coding exercises and real-world problem-solving, to reinforce core linear algebra concepts. We also utilized active learning techniques, such as collaborative group worksheets, to promote a more engaging and interactive learning experience.

Our efforts were well received by students, who recognized the practical applications of linear algebra and felt better prepared for subsequent courses in their curricula. These positive outcomes encouraged us to continue exploring new pedagogical techniques and expand our approach to other courses and disciplines.

Looking ahead, our team plans to develop a practical guide for teaching linear algebra with MATLAB. This guide will be designed to help educators incorporate computational components into their teaching methods effectively. It will also showcase the practical applications of linear algebra and highlight its significance as a scientific tool in various fields. Our ultimate aim is to provide students with a comprehensive understanding of linear algebra and equip them with the skills to apply this knowledge in real-world situations.

In summary, our project was successful in elevating the traditional linear algebra course at UVA by promoting an active, engaging, and practical learning environment. We believe that our approach has the potential to transform the way linear algebra is taught and learned, and we look forward to continuing our efforts to enhance STEM education at UVA and beyond.

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In conclusion, I am grateful for the opportunity to undertake this project, and I am proud of the results we have achieved. This project has the potential to revolutionize the way linear algebra is taught, and I am excited to see its impact on future generations of students.

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